The explicit calculation of Čech cohomology and an extension of Davenport's inequality

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Abstract

We extend Davenport's inequality to general elliptic curves over $\mathbb{C}(t)$ written in Weierstrass forms. The obtained result is an effective version of a result by Voloch, and also improves a bound given by Hindry-Silverman. The method depends on an explicit calculation of the Čech cohomology of sheaves of differentials on an elliptic surface.

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1 Introduction

Let $a, b \in \mathbb{C}[t]$ be polynomials. The Weierstrass equation $y^2 = 4x^3 - 3ax + b$ defines an elliptic curve $E/\mathbb{C}(t)$, and for a place v of $\mathbb{C}(t)$, we can give lower bounds of the valuation $v(r)$ for every rational points $(x = r, y = s) \in E(\mathbb{C}(t))$.

The existence of such a lower bound is proved by Manin[13, Theorem 4]:

Proposition 1 (Manin)**.** *Let C be an algebraic curve over $\mathbb{C}$, let $K = k(C)$ be its function field, and let $E : y^2 = 4x^3 - 3ax + b$ $(a, b \in K)$ be an elliptic curve over K. Fix a $p \in C$, then there exists an integer $C_{a,b,p}$ such that for all $(r, s) \in E(K)$ one has $v_p(r) \geq C_{a,b,p}$.*

As pointed out by Manin[13, §11], Proposition 1 is a strong statement which easily implies Siegel's theorem on the finiteness of integral points on an affine elliptic curve defined over a function field. In 1994, Voloch[22] noted that the following lemma together with the Mordell-Weil theorem gives a short proof of Proposition 1 (also cf. [18, Chap.III §12]).

Lemma 1. *Using the notation from Proposition 1, put $E_i = \{(r, s) \in E(K); v_p(r) \leq -2i\}$. Then E_i is a subgroup of $E(C)$, $E_i \supset E_{i+1}$, and E_i/E_{i+1} is torsion-free.*

However, Voloch's proof is not effective; we can ask for an explicit lower bound of $v_p(r)$ and a nontrivial upper bound of $\operatorname{rank} E_i$.

We restrict to the case $C = \mathbf{P}^1_{\mathbb{C}}$. Without losing genericity, we only consider the valuation v_∞ at $t = \infty$. And to be more intuitive, we will use the notation 'deg' to denote

$-v_\infty$, this is the degree of a polynomial, and for $f = P/Q$ where P, Q are polynomials, $\deg f$ is equal to $\deg P - \deg Q$.

Then we can state the questions as:

Question 1. Let $E/\mathbb{C}(t)$ be an elliptic curve defined by $y^2 = 4x^3 - 3ax + b$ $(a, b \in \mathbb{C}(t))$. Give an upper bound of $\frac{1}{2}\deg r$ for every $(r, s) \in E(\mathbb{C}(t))$, using only the coefficients a, b in the Weierstrass equation.

Question 2. Give an upper bound of $\operatorname{rank} E_i$ where $E_i = \{(r, s) \in E(\mathbb{C}(t)); \frac{1}{2}\deg r \geq i\}$.

The main theorem of this paper will give an answer for these two questions.

As for Question 1, some results are known for *integral* points $(r, s) \in E(\mathbb{C}(t))$ where $r, s \in \mathbb{C}[t]$. The first one is Davenport's inequality[2] in 1965:

Proposition 2 (Davenport)**.** *Let $f, g \in \mathbb{C}[t]$ be polynomials and $f^3 - g^2 \neq 0$. Then $\frac{1}{2}\deg f \leq \deg(f^3 - g^2) - 1$.*

Viewed as a proposition on elliptic curves over $\mathbb{C}(t)$, Proposition 2 states

$$\frac{1}{2}\deg r \leq \deg h - 1 \tag{1}$$

for any $(r, s) \in E(\mathbb{C}(t))$ where $r, s \in \mathbb{C}[t]$ and $E : y^2 = x^3 + h$ $(0 \neq h \in \mathbb{C}[t])$.

Stothers[20] gave a characterization of the polynomials which satisfy the equality, namely if $\frac{1}{2}\deg f = \deg(f^3 - g^2) - 1$, then $f^3/(f^3 - g^2)$ is a Belyi function (also cf. [9, §2.5]). And the story has been generalized by Zannier[23].

The same method used in their proof (*i.e.* the Riemann-Hurwitz) also can be used to give bounds for solutions of unit equations, and has been applied to general hyperelliptic curves over function fields by Mason[14], Schmidt[17], and Hindry-Silverman [8]. Restricted to the case of elliptic curves over $\mathbb{C}(t)$, [8, Proposition 8.2] can be rephrased as:

Proposition 3 (Hindry-Silverman)**.** *Let $E : y^2 = 4x^3 - 3ax + b$ be an elliptic curve over $\mathbb{C}(t)$ where $a, b \in \mathbb{C}[t]$ and $\Delta := a^3 - b^2$ is nonzero. Then for any $(r, s) \in E(\mathbb{C}(t))$ where $r, s \in \mathbb{C}[t, \Delta^{-1}]$, we have* $\mathrm{DEG}(s^4/\Delta) \leq 24(N_0(\Delta) - 1)$.

Here $N_0(\Delta)$ denotes the number of distinct zeros of Δ. For a rational function f, $\mathrm{DEG}(f)$ is regarded as the degree of the field extension $[\mathbb{C}(t) : \mathbb{C}(f)]$, *i.e.* the degree of the map to $\mathbf{P}^1_{\mathbb{C}}$ defined by f. So obviously $\deg f \leq \mathrm{DEG}(f)$.

Proposition 3 immediately implies that

$$\frac{1}{2}\deg r \leq 2N_0(\Delta) - 2 + \frac{1}{12}\deg\Delta \tag{2}$$

for any $(r, s) \in E(\mathbb{C}(t))$ where $r, s \in \mathbb{C}[t, \Delta^{-1}]$ and $E : y^2 = 4x^3 - 3ax + b$ $(a, b \in \mathbb{C}[t])$.

To state the main theorem of this paper, we use the following notations:

Notation 1. Let the elliptic curve $E/\mathbb{C}[t]$ be defined by a Weierstrass equation $y^2 = 4x^3 - 3ax + b$ $(a, b \in \mathbb{C}[t])$.

- Let n be the least integer such that $\deg a \leq 4n, \deg b \leq 6n$.

- Put $\Delta = a^3 - b^2$, $\Lambda = 2b'a - 3ba'$, $\Phi = \frac{a\Lambda^2 - (\Delta')^2}{12\Delta} = \frac{1}{3}(b')^2 - \frac{3}{4}a(a')^2$.
- Put $P_2 = \Delta\Lambda$, $P_1 = \Delta'\Lambda - \Delta\Lambda'$, $P_0 = \frac{1}{12}(\Delta''\Lambda - \Delta'\Lambda' + \Phi\Lambda)$.
- For $g, h \in \mathbb{C}(t)$, put $\rho(g,h) = \Delta\Lambda h' - \Delta\Lambda' h + \frac{11}{12}\Delta'\Lambda h - 6(P_2 g'' + P_1 g' + P_0 g)$.
- For $P, Q \in \mathbb{C}[t]$, let $\lfloor P/Q \rfloor$ and $(P \text{ MOD } Q)$ be the polynomials such that $P = \lfloor P/Q \rfloor Q + P \text{ MOD } Q$ and $\deg(P \text{ MOD } Q) < \deg Q$.
- Put $B = \{\rho(p,q) \text{ MOD } \Lambda^2; p, q \in \mathbb{C}[t]\}$.

And we make the following assumptions:

- Assume the equation is minimal, *i.e.* there is no nonconstant polynomial l such that a is divisible by l^4 and b is divisible by l^6.
- Assume $\Lambda \neq 0$. This is to say that the J-invariant of E is nonconstant.

Now the main theorem states:

Theorem 1. *Using Notation 1, we have:*

1. *Put* $c = \min\{\deg\beta; \beta \in B, \beta \neq 0\}$. *Then for any* $(r,s) \in E(\mathbb{C}(t))$,
$$\frac{1}{2}\deg r \leq \deg(\Delta\Lambda) - c - 2 \tag{3}$$
2. *Put* $E_i = \{(r,s) \in E(\mathbb{C}(t)); \frac{1}{2}\deg r \geq i\}$. *Then:*
   (a) *For* $c \leq j \leq 2\deg\Lambda - 1$,
$$\operatorname{rank} E_{\deg(\Delta\Lambda)-j-2} \leq \dim_{\mathbb{C}}\{\beta \in B; \deg\beta \leq j\}.$$
   (b) *For* $2\deg\Lambda + n - 2 \leq k$,
$$\operatorname{rank} E_{\deg(\Delta\Lambda)-k-2} \leq \dim_{\mathbb{C}} B + k - (2\deg\Lambda + n - 2).$$

Example 1. Consider a 'general case', where $\deg\Delta = 12n$ and $\gcd(\Delta, \Delta') = 1$. That is to say, all singular fibers of E is of type I_1 and the ∞-fiber is not singular. In this case, formula (2) gives an inequality $\frac{1}{2}\deg r \leq 25n - 2$, while (3) implies an inequality $\frac{1}{2}\deg r \leq 22n - 4$, since $c \geq 0$ and $\deg\Lambda \leq 10n - 2$. So 'in general' (3) will give a better inequality than (2).

Example 2. Consider a special case where a is a *constant*. Then we can verify that $\rho(p,q) \text{ MOD } \Lambda^2 = \Lambda f' - \Lambda' f$ where $f = (-\frac{1}{2}\Delta' p - 6p'\Delta + q\Delta) \text{ MOD } \Lambda$. So

1. $c = \deg\Lambda + \deg\gcd(\Delta, \Delta') - 1$ and for any $(r,s) \in E(\mathbb{C}(t))$,
$$\frac{1}{2}\deg r \leq N_0(\Delta) - 1. \tag{4}$$
2. We have:

(a) For $0 \leq i \leq \deg \Lambda - \deg \gcd(\Delta, \Delta')$, $\operatorname{rank} E_{N_0(\Delta)-i} \leq i$.

(b) For $j \geq \deg \Lambda - \deg \gcd(\Delta, \Delta')$, $\operatorname{rank} E_{N_0(\Delta)-n-j} \leq j$.

Example 3. Consider the equation $E: y^2 = 4x^3 + 108x + 81t^2$. It has a solution $(x = t^6 + 6t^2, y = 2t^9 + 18t^5 + 27t)$ which shows that the inequality (4) is tight.

Note that the form of the inequality (4) coincides with Davenport's inequality (1) if we set $a = 0$. And Example 3 shows that this is also a tight inequality as Davenport's inequality is. It may be interesting to ask that if there is a brief characterization of those examples which satisfy the equality, like the characterization done by Stothers?

The method used by Hindry-Silverman originates from Siegel's reduction, which reduces the problem of integral points of elliptic curves to the problem of solutions of unit equations. It applies to integral points only. Our approach however is near to Manin's method using Gauss-Manin connection and Manin's map. We will see that the inequality (3) comes from the estimation of the degrees of the polynomials in the image of a $\mathbb{C}$-linear map $H^1_{prim}(\widetilde{E}, \Omega^1_{\widetilde{E}}) \oplus H^0(\widetilde{E}, \Omega^2_{\widetilde{E}}) \to \mathbb{C}[t]$, whose restriction on the Mordell-Weil lattice[19] $MWL \subset H^1_{prim}(\widetilde{E}, \Omega^1_{\widetilde{E}})$ coincides with Manin's map $MWL \hookrightarrow \mathbb{C}[t]$. So, if in some situation we can get a even better inequality than (3), it will also give some nontrivial restrictions on how the Mordell-Weil lattice can be emmbedded into $H^1_{prim}(\widetilde{E}, \Omega^1_{\widetilde{E}})$.

In order to illustrate the idea of the proof, it may be helpful to give here some investigation on the much simpler equation $E: y^2 = x^3 + h$ $(0 \neq h \in \mathbb{C}[t])$. I would like to begin with an extremely simple proof of Davenport's inequality:

Proof of Proposition 2. For any $s(t)^2 = r(t)^3 + h(t)$, we have

$$\begin{aligned} \frac{hr' - \frac{1}{3}h'r}{s} &= \frac{(s^2 - r^3)r' - \frac{1}{3}h'r}{s} \\ &= \frac{s^2 r' - \frac{1}{3}r(3r^2 r' + h')}{s} \\ &= sr' - \frac{2}{3}rs' \end{aligned} \tag{5}$$

Now assume $\deg r > \frac{1}{3}\deg h$, then $\deg(hr' - \frac{1}{3}h'r) = \deg r + \deg h - 1$ and $\deg s = \frac{3}{2}\deg r$. The left hand side of the above equality has a degree $\deg h - 1 - \frac{1}{2}\deg r$, and the right hand side is a polynomial so has a degree ≥ 0. Hence $\deg h - 1 - \frac{1}{2}\deg r \geq 0$ or $\frac{1}{2}\deg r \leq \deg h - 1$. $\square$

The interpretation is that the magical expression $\frac{hr' - \frac{1}{3}h'r}{s}$ in (5) comes out from Manin's map and has a cohomological meaning. We can calculate

$$\frac{\partial}{\partial t}\frac{dx}{\sqrt{x^3 + h(t)}} = -\frac{1}{2}\frac{h'dx}{(x^3 + h)^{\frac{3}{2}}}$$

and

$$d\left(\frac{x}{\sqrt{x^3 + h}}\right) = -\frac{1}{2}\frac{dx}{\sqrt{x^3 + h}} + \frac{3}{2}h\frac{dx}{(x^3 + h)^{\frac{3}{2}}},$$

so

$$(h\frac{\partial}{\partial t}+\frac{1}{6}h')\frac{dx}{y}=d(-\frac{1}{3}h'\frac{x}{y}),$$

thus for any $(r(t),s(t))\in E(\mathbb{C}(t))$ of $E:y^2=x^3+h$ we have

$$\begin{aligned}(h\frac{\partial}{\partial t}+\frac{1}{6}h')\int_\infty^{r(t)}\frac{dx}{y}&=h\frac{r'}{s}+\int_\infty^r(h\frac{\partial}{\partial t}+\frac{1}{6}h')\frac{dx}{y}\\&=h\frac{r'}{s}+\int_\infty^r d(-\frac{1}{3}h'\frac{x}{y})\\&=h\frac{r'}{s}-\frac{1}{3}h'\frac{r}{s}\end{aligned}\tag{6}$$

which is the content of expression (5). We know $(h\frac{\partial}{\partial t}+\frac{1}{6}h')\int_\infty^r\frac{dx}{y}$ is an entire function divisible by $\gcd(h,h')$, and at the same time also a rational function, so it must be a polynomial of degree$\geq \deg\gcd(h,h')$. This way we get a slightly stronger version of Davenport's inequality:

Proposition 4. *Let $E:y^2=x^3+h$ be given by a minimal equation. Then for any $(r,s)\in E(\mathbb{C}(t))$ we have $\frac{1}{2}\deg r\leq N_0(h)-1$.*

Proof. Since the equation is minimal we have $\frac{1}{6}\deg h\leq N_0(h)-1$. Then the same argument as in the proof of Proposition 2, comparing the degree of the two sides of (6) instead of (5), gives the statement. □

Now regard $E:y^2=x^3+h(t)$ as a smooth proper elliptic fibration over the affine curve $A=\mathbf{Spec}\,\mathbb{C}[t,h^{-1}]$, denote the 0-section of E by $\mathfrak{o}$, the normal bundle of $\mathfrak{o}$ by $\mathscr{N}$. The J-invariant of E is constant, so $\mathscr{N}$ is in fact a locally constant line bundle, we denote the associated locally constant sheaf by $\mathbb{C}_{\mathscr{N}}$. The relative 1-form $\frac{dx}{y}$ can be viewed as a nonvanishing section of the dual bundle of $\mathscr{N}$, so its dual $(\frac{dx}{y})^*$ is a nonvanishing section of $\mathscr{N}$. Then it is not hard to realize that the expression $\{(h\frac{\partial}{\partial t}+\frac{1}{6}h')\int_\infty^r\frac{dx}{y}\}dt(\frac{dx}{y})^*$ for a section $\mathfrak{s}=(r,s)\in E(\mathbb{C}(t))$ represents an element in the cohomology group $H^1(A_{an},\mathbb{C}_{\mathscr{N}})$ corresponding to $\mathfrak{s}-\mathfrak{o}$. (Here A_{an} is the associated complex analytic space of A.) Elements corresponding to some $\mathfrak{s}-\mathfrak{o}$ form a lattice of a subspace of $H^1(A_{an},\mathbb{C}_{\mathscr{N}})$, on the other hand $(h\frac{\partial}{\partial t}+\frac{1}{6}h')\int_\infty^r\frac{dx}{y}$ is a polynomial divisible by $\gcd(h,h')$, and the vector space of such polynomials whose degree$\leq\deg(\gcd(h,h'))+i-1$ only has a $\mathbb{C}$-dimension i, so we conclude:

Proposition 5. *Let $E:y^2=x^3+h$ be given by a minimal equation. Then put $E_i=\{(r,s)\in E(\mathbb{C}(t));\frac{1}{2}\deg r\geq N_0(h)-i\}$ we have* $\operatorname{rank}E_i\leq 2i$.

To extend the above story to general elliptic curves over $\mathbb{C}(t)$, we should use a differential operator of order 2 instead of the operator $(h\frac{\partial}{\partial t}+\frac{1}{6}h')$, since the J-invariant is no longer constant. The proof of the main theorem follows the steps:

1. Find the differential operator P which annihilates the relative 1-form $\frac{dx}{y}$.

2. For $(r,s)\in E(\mathbb{C}(t))$, calculate the degree of $P\int_\infty^r\frac{dx}{y}$.

3. Find a cohomological interpretation of the expression $P\int_\infty^r \frac{dx}{y}$.

4. Explicitly calculate the cohomology of E.

Step 1 and 2 are elementary calculus. We will relate the expression $P\int_\infty^r \frac{dx}{y}$ to the algebraic de Rham cohomology of E viewed as an elliptic surface, and deal with it via an explicit calculation of the Čech cohomology of sheaves of differentials.

It seems to me that the method using explicit calculation of Čech cohomology is not yet widely used, so the article also intends to be a summary and introduction to such calculations. It may be thought as difficult when dealing with Čech cohomology, that the localization A_f of a ring A along f is generally not finitely generated as an A-module. We avoid this problem by connecting the Čech complex to a finitely generated complex via a bicomplex. This idea is near the one called "eyeballing" in [4], however the main concern of [4] is on the computation of the $\bigoplus_n H^0(\mathscr{O}_{\mathbf{P}^r}(n))$-module structure of $\bigoplus_n H^i(\mathscr{F}(n))$ for a coherent sheaf $\mathscr{F}$ on $\mathbf{P}^r$, explicit computation of Čech cohomology is not considered there. On the other hand it is noted in [16] that localization is finitely generated when viewed as a D-module, so we can deal with it using gröbner basis for Weyl algebras. Some difficulty coming from the non-commutativity should be overcome, and an algorithm to calculate the de Rham cohomology of the complement of an affine variety is given in [16]. The methods used in this paper however take another approach.

The paper is organized into 7 sections. In §2 we will briefly review the formalization of Gauss-Manin connection and Manin's map, then step 1 and step 2 mentioned above will be done. In §3 we will discuss the relationship between Manin's map and the algebraic de Rham cohomology, which forms the very essence of this paper. In §4 we prove the main theorem, leaving a calculation result on the algebraic de Rham cohomology of the elliptic surface unproved, and this key lemma will be proved in §7, after some generic consideration in §5 on how the Čech cohomology of a coherent sheaf of a projective scheme can be calculated, and in §6 how the free resolutions of the sheaves of differentials of a hypersurface can be constructed. In §7 we do the actual calculation, which completes the proof and also serves as an example of §5 and §6.

2 Gauss-Manin connection and Manin's map

In this section, we will reveal that the polynomials P_2, P_1, P_0 in Notation 1 are coefficients of the Picard-Fuchs equation of the elliptic curve $E/\mathbb{C}(t)$. Then we note the important fact that in almost cases, for a section $\mathfrak{s} = (r,s) \in E(\mathbb{C}(t))$, the degree of Manin's map $\deg \mu(\mathfrak{s})$ is related to $\deg r$ by $\deg \mu(\mathfrak{s}) = -\frac{1}{2}\deg r + \deg(\Delta\Lambda) - 2$.

We begin with a review on Gauss-Manin connection and Manin's map, following the formalization in [13]. Let K be a function field over $\mathbb{C}$ and endowed with a derivation ∂ (In our case, $K = \mathbb{C}(t)$ and $\partial = \frac{\partial}{\partial t}$), L a function field over K (In our case $L = K(x,y), y^2 = 4x^3 - 3ax + b$) and for simplicity assume that L/K is of transcendence degree 1. Furthermore assume that there exists a K-rational point on the curve L/K. Choose a transcendence base $x \in L$ (In our case we choose $x \in K(x,y)$). Denote by ∂_x the unique derivation of L which extends ∂ and satisfies $\partial_x x = 0$. Let $\Omega_{L/K}$ be the L-module of relative 1-forms. Then ∂_x acts on $\Omega_{L/K}$ by $\partial_x(udx) = (\partial_x u)dx$. Let

$B, Z \subset \Omega_{L/K}$ be the K-subspace of exact and closed forms, respectively. Since we have assumed that L/K is of transcendence degree 1, in this case we have $Z = \Omega_{L/K}$ and $B = d(L)$. Then the induced action of ∂_x on Z/B turns out to be independent of the choice of the transcendence base x. Thus Z/B can be viewed as a $K[\partial]$-module, the action of ∂ is called the *Gauss-Manin connection*.

Let $(\omega_1, \dots, \omega_g)$ be a K-basis of the relative 1-forms of the first kind of L/K (Here g is the genus of L/K, and in our case $g = 1$). Denote by $\bar{\omega}_i$ the classes of ω_i in Z/B. A *Picard-Fuchs equation* $\mathscr{P}$ is any relation of the form

$$\mathscr{P} : \sum_{i=1}^{g} P_i \bar{\omega}_i = 0,\ P_i \in K[\partial].$$

And if we choose a transcendence base $x \in L$ of L/K, $\mathscr{P}$ has a *representation* in the form

$$\sum_{i=1}^{g} P_{ix} \omega_i = dz_x,\ P_{ix} \in K[\partial_x], z_x \in L.$$

The set of all Picard-Fuchs equations is a submodule of the left $K[\partial]$-module $K[\partial]^{\oplus g}$, thus is finitely generated. This module, up to isomorphism, obviously does not depend on the choice of the K-basis $(\omega_1, \dots, \omega_g)$. In our case $g = 1$, the module of all Picard-Fuchs equations is a left ideal of $K[\partial]$, with respect to a chosen relative 1-form ω of the first kind (In our case the relative 1-form $\frac{dx}{y}$ is chosen). Any left ideal of $K[\partial]$ is a principle ideal. When assuming that the J-invariant is not constant, this generator should be an operator of order 2. So there should be no ambiguity, up to a multiple of K, for us to indicate *the* Picard-Fuchs equation of order 2.

Now let $\kappa : V \to C$ be a model of L/K, and let $\mathfrak{o}$ be a fixed K-rational point of V. For any Picard-Fuchs equation $\mathscr{P} : \sum_{i=1}^{g} P_i \bar{\omega}_i = 0$ with respect to a K-basis $(\omega_1, \dots, \omega_g)$ of the relative 1-forms of the first kind, *Manin's map* $\mu_{\mathscr{P}}$ assign an element of K to every K-rational point $\mathfrak{s}$ of V, namely

$$\mu_{\mathscr{P}}(\mathfrak{s}) = \sum_{i=1}^{g} P_i \int_{\mathfrak{o}}^{\mathfrak{s}} \bar{\omega}_i$$

which can be perfectly defined using only an algebraic language. In the $g = 1$ case, we can omit the suffix $\mathscr{P}$ and always regard the Picard-Fuchs equation as the generator of order 2. In this case choosing a transcendence base x of L/K and a relative 1-form udx of the first kind, the Picard-Fuchs equation has a representation

$$(\partial_x^2 + a\partial_x + b)udx = dw,\ a, b \in K, w \in L$$

and we can choose w to be such that $w_{\mathfrak{o}}$, the value of w at $\mathfrak{o}$, is 0. Then for any K-point $\mathfrak{s}$ we have

$$\mu(\mathfrak{s}) = w_{\mathfrak{s}} + au_{\mathfrak{s}}\partial x_{\mathfrak{s}} + (\partial_x u)_{\mathfrak{s}}\partial v_{\mathfrak{s}} + \partial(u_{\mathfrak{s}}\partial x_{\mathfrak{s}}).$$

The main point of Manin's map is that it transforms the rather intractable additions of Abelian integrals in the Jacobian variety into additions of K. When considering about its relation with cohomologies, at a first eye it seems that there is no reason for us to take only relative 1-forms of the first kind. In fact we can take relative 1-forms $\eta_1,\dots,\eta_g$ of the second kind (*i.e.* meromorphic differentials with no residues) such that $(\bar{\omega}_1,\dots,\bar{\omega}_g,\bar{\eta}_1,\dots,\bar{\eta}_g)$ forms a basis of $H^1(\mathfrak{f},\mathbb{C})$ for generic fiber $\mathfrak{f}$ of κ. Then the action of the Gauss-Manin connection ∂ can be restricted to the K-subspace $W\subset Z/B$ generated by $(\bar{\omega}_1,\dots,\bar{\omega}_g,\bar{\eta}_1,\dots,\bar{\eta}_g)$. So $\partial(W)\subset W$ and there is a matrix A with coefficients in K such that

$$\partial\begin{pmatrix}\bar{\eta}_1 & \cdots & \bar{\eta}_g & \bar{\omega}_1 & \cdots & \bar{\omega}_g\end{pmatrix}=\begin{pmatrix}\bar{\eta}_1 & \cdots & \bar{\eta}_g & \bar{\omega}_1 & \cdots & \bar{\omega}_g\end{pmatrix}A.$$

A Picard-Fuchs equation is just a relation obtained from eliminating $\bar{\eta}_i$ in the above system of relations of order 1, and the $\mathbb{C}$-vector space $W/\partial(W)$ has a natural map to the cohomology group $\varinjlim_U H^1(U_{an},R^1\kappa_*\mathbb{C})$, where $U\subset C$ runs over all Zariski open sets of the base curve C.

Nevertheless, as we will see in the next section, the effect of using a relative 1-form of the first kind turns out to be clear, when considering with algebraic de Rham cohomology. In the remaining of this section we will actually calculate the Picard-Fuchs equation, Manin's map and its degree.

Lemma 2. *Using Notation 1, put $K=\mathbb{C}(t)$ and $L=K(x,y)$. Let ∂_x be the extension of $\frac{\partial}{\partial t}$ to L such that $\partial_x x=0$, and let $d:L\to\Omega_{L/K}$ be the relative differential. Denote the equivalence relation in $\Omega_{L/K}/d(L)$ by '$\cong$'. Then*

$$(P_2\partial_x\partial_x+P_1\partial_x+P_0)\frac{dx}{y}\cong 0.$$

Proof. Put $\omega=\frac{dx}{y}$. Since $d(\frac{x}{y})=-3a\frac{xdx}{y^3}+\frac{3}{2}b\frac{dx}{y^3}-\frac{1}{2}\frac{dx}{y}$ we have

$$\omega\cong -6a\frac{xdx}{y^3}+3b\frac{dx}{y^3}.$$

Now

$$\partial_x\omega=\frac{3}{2}a'\frac{xdx}{y^3}-\frac{1}{2}b'\frac{dx}{y^3}.$$

Put

$$\eta=6b\frac{xdx}{y^3}-3a^2\frac{dx}{y^3}.$$

Then it is easy to see that

$$12\Delta\partial_x\omega\cong\Lambda\eta-\Delta'\omega. \tag{7}$$

Using $d(\frac{1}{y^3})=-18\frac{x^2dx}{y^5}+\frac{9}{2}a\frac{dx}{y^5}$ we have

$$\frac{x^2dx}{y^5}\cong\frac{1}{4}a\frac{dx}{y^5}.$$

And using $d(\frac{x}{y^3})=-9a\frac{xdx}{y^5}+\frac{9}{2}b\frac{dx}{y^5}-\frac{7}{2}\frac{dx}{y^3}$ we have

$$a\frac{xdx}{y^5}\cong-\frac{1}{2}b\frac{dx}{y^5}+\frac{7}{18}\frac{dx}{y^3}.$$

From $d(\frac{x^2}{y^3}) = -9a\frac{x^2dx}{y^5} + \frac{9}{2}b\frac{xdx}{y^5} - \frac{5}{2}\frac{xdx}{y^3}$ we get

$$b\frac{xdx}{y^5} \cong 2a\frac{x^2dx}{y^5} + \frac{5}{9}\frac{xdx}{y^3} \cong \frac{1}{2}a^2\frac{dx}{y^5} + \frac{5}{9}\frac{xdx}{y^3}.$$

So

$$\begin{aligned}\partial_x\eta &= 27ba'\frac{x^2dx}{y^5} - (9bb' + \frac{27}{2}a^2a')\frac{xdx}{y^5} + \frac{9}{2}a^2b'\frac{dx}{y^5} + 6b'\frac{xdx}{y^3} - 6aa'\frac{dx}{y^3} \\ &\cong b'\frac{xdx}{y^3} - \frac{3}{4}aa'\frac{dx}{y^3}\end{aligned}$$

Then we can check that

$$12\Delta\partial_x\eta \cong \Delta'\eta - a\Lambda\omega. \tag{8}$$

So the statement is proved by eliminating η from (7) and (8). $\square$

It turns out that the operator $P_2\frac{\partial}{\partial t}\frac{\partial}{\partial t} + P_1\frac{\partial}{\partial t} + P_0$ almost preserves the degree.

Lemma 3. *Let g be a meromorphic function on a neighborhood of $t = \infty$. Then in the following cases* $\deg(P_2g'' + P_1g' + P_0g) < \deg(\Delta\Lambda) + \deg g - 2$*:*

- *The ∞-fiber of E is nonsingular and* $\deg g = -n$
- *The ∞-fiber of E is nonsingular and* $\deg g = -11n + \deg\Lambda + 1$
- *The ∞-fiber of E is of type I_m $(m \geq 1)$ and* $\deg g = -n$

Otherwise we have $\deg(P_2g'' + P_1g' + P_0g) = \deg(\Delta\Lambda) + \deg g - 2$.

Proof. Denote the leading coefficients of Δ, Λ, g by c_Δ, c_Λ, c_g, respectively. According to the definition of P_2, P_1, P_0, we will first reveal some relations among $\deg\Delta$, $\deg\Lambda$ and $\deg\Phi$.

(i) From the identity $a^2\Lambda = 2b'\Delta - b\Delta'$ and $b\Lambda = 3a'\Delta - a\Delta'$ we get

$$\begin{aligned}\deg\Lambda &\leq \deg\Delta - 1 + \deg b - 2\deg a \\ \deg\Lambda &\leq \deg\Delta - 1 + \deg a - \deg b\end{aligned}$$

Then eliminate $\deg a$ and $\deg b$ respectively, we get

$$\begin{aligned}\deg\Lambda &\leq \deg\Delta - 1 - \frac{1}{3}\deg b \\ \deg\Lambda &\leq \deg\Delta - 1 - \frac{1}{2}\deg a\end{aligned}$$

Now by the definition of n we have either $\deg a \geq 4n - 3$ or $\deg b \geq 6n - 5$, any case the inequalities above will imply

$$\deg\Lambda \leq \deg\Delta - 2n \tag{9}$$

and the equality holds if and only if $\deg a = 4n - 2$, $\deg b = 6n - 3$ and $\deg\Delta \neq 12n - 6$. This is to say that the ∞-fiber is of type I_m^* $(m \geq 1)$. For the notation of singular fiber types of elliptic fibrations, cf. [12] or [18, Chap.IV §8].

(ii) Using the definition $\Phi = \frac{a\Lambda^2-(\Delta')^2}{12\Delta}$, by (9) we see that $\deg \Phi \leq \deg \Delta - 2$, and if $\deg(a\Lambda^2) < 2\deg(\Delta')$, then the coefficient of Φ at degree $\deg \Delta - 2$ is simply

$$-\frac{1}{12}(\deg \Delta)^2 c_\Delta.$$

On the otherhand we have $\deg(a\Lambda^2) = 2\deg(\Delta')$ in the following cases:

(a) $\deg a = 4n-2$, $\deg b = 6n-3$, $\deg \Delta \neq 12n-6$ and $\deg \Lambda = \deg \Delta - 2n$, the ∞-fiber is of type I_m^* $(m \geq 1)$:
In this case, from the identity

$$a\Lambda^2 = (2\frac{b'}{b}\Delta - \Delta')(3\frac{a'}{a}\Delta - \Delta'),$$

we see that the coefficient of Φ at degree $\deg \Delta - 2$ is

$$\begin{aligned} &\frac{1}{12}\{(2\deg b - \deg \Delta)(3\deg a - \deg \Delta) - (\deg \Delta)^2\}c_\Delta \\ =&\frac{1}{12}\{(12n-6-\deg \Delta)^2 - (\deg \Delta)^2\}c_\Delta \end{aligned}$$

(b) $\deg \Lambda = \deg \Delta - 2n - 1$ and $\deg a = 4n$:
In this case $\deg \Delta \neq 12n$ since $\deg \Lambda \leq 10n-2$. By $b\Lambda = 3a'\Delta - a\Delta'$ we get $\deg b = 6n$. This is to say that the ∞-fiber is of type I_m $(m \geq 1)$. Now similar to (a) we can calculate the coefficient of Φ at degree $\deg \Delta - 2$ to be

$$\frac{1}{12}\{(12n - \deg \Delta)^2 - (\deg \Delta)^2\}c_\Delta.$$

(iii) From (ii) we see that $\deg P_0 \leq \deg(\Delta\Lambda) - 2$, thus

$$\deg(P_2 g'' + P_1 g' + P_0 g) \leq \deg(\Delta\Lambda) + \deg g - 2.$$

And if $\deg(a\Lambda^2) < 2\deg(\Delta')$ *i.e.* the coefficient of Φ at degree $\deg \Delta - 2$ is $-\frac{1}{12}(\deg \Delta)^2 c_\Delta$, we can calculate the coeffient of $P_2 g'' + P_1 g' + P_0 g$ at degree $\deg(\Delta\Lambda) + \deg g - 2$ to be

$$(\deg g + \frac{1}{12}\deg \Delta)(\deg g + \frac{11}{12}\deg \Delta - \deg \Lambda - 1)c_\Delta c_\Lambda c_g.$$

Otherwise,

(a) If the ∞-fiber is of type I_m^* $(m \geq 1)$, the coeffient of $P_2 g'' + P_1 g' + P_0 g$ at degree $\deg \Delta\Lambda + \deg g - 2$ is

$$(\deg g + n - \frac{1}{2})^2 c_\Delta c_\Lambda c_g.$$

(b) If the ∞-fiber is of type I_m $(m \geq 1)$, the coeffient of $P_2 g'' + P_1 g' + P_0 g$ at degree $\deg \Delta\Lambda + \deg g - 2$ is

$$(\deg g + n)^2 c_\Delta c_\Lambda c_g.$$

So the possibilities for this coefficient to be 0 are listed in the statement. $\square$

We can also similarly prove the following, which will be used later:

Lemma 4. *Fix a $w \in \mathbb{C}$. For a function g meromorphic on a neighborhood of $t = w$, denote the order of zeros of g at $t = w$ by $\operatorname{ord}_w g$. In the following cases we have* $\operatorname{ord}_w(P_2 g'' + P_1 g' + P_0 g) > \operatorname{ord}_w(\Delta\Lambda) + \operatorname{ord}_w g - 2$*:*

- *The w-fiber of E is nonsingular and* $\operatorname{ord}_w g = 0$
- *The w-fiber of E is nonsingular and* $\operatorname{ord}_w g = \operatorname{ord}_w \Lambda + 1$
- *The w-fiber of E is of type I_m $(m \geq 1)$ and* $\operatorname{ord}_w g = 0$

Otherwise we have $\operatorname{ord}_w(P_2 g'' + P_1 g' + P_0 g) = \operatorname{ord}_w(\Delta\Lambda) + \operatorname{ord}_w g - 2$.

Proof. Totally parallel to the proof of Lemma 3. Notice that the Weierstrass equation is minimal, so we have either $\operatorname{ord}_w a \leq 3$ or $\operatorname{ord}_w b \leq 5$, and $\operatorname{ord}_w \Lambda \geq \operatorname{ord}_w \Delta - 2$. $\square$

Now by Lemma 2, we can take the Picard-Fuchs equation to be $P_2 \frac{\partial}{\partial t}\frac{\partial}{\partial t} + P_1 \frac{\partial}{\partial t} + P_0$ with respect to $\frac{dx}{y}$. Then for $\mathfrak{s} = (r,s) \in E(K)$, Manin's map is by definition

$$\mu(\mathfrak{s}) = (P_2 \frac{\partial}{\partial t}\frac{\partial}{\partial t} + P_1 \frac{\partial}{\partial t} + P_0) \int_{\mathfrak{o}}^{\mathfrak{s}} \frac{dx}{y}.$$

Proposition 6. *For a section $\mathfrak{s} = (r,s) \in E(\mathbb{C}(t))$, assume $\frac{1}{2}\deg r > n$. Then we have $\deg \mu(\mathfrak{s}) \leq 2\deg\Lambda + n - 2$ if $\deg\Delta = 12n$ and $\frac{1}{2}\deg r = 11n - \deg\Lambda - 1$, otherwise $\deg \mu(\mathfrak{s}) = -\frac{1}{2}\deg r + \deg(\Delta\Lambda) - 2$.*

Proof. The ∞-model of E is $E_\infty : y_\infty^2 = 4x_\infty^3 - 3a_\infty x_\infty + b_\infty$ where

$$x_\infty = t^{-2n}x,\ y_\infty = t^{-3n}y,\ a_\infty = t^{-4n}a,\ b_\infty = t^{-6n}b.$$

For a section $\mathfrak{s} = (r,s) \in E(\mathbb{C}(t))$, if $\frac{1}{2}\deg r > n$ then $\mathfrak{s}$ intersects the 0-section $\mathfrak{o}$ at $t = \infty$ with a multiplicity $\frac{1}{2}\deg r - n$, which means that we can choose a branch of the multivalued function $\int_{\mathfrak{o}}^{\mathfrak{s}} \frac{dx_\infty}{y_\infty}$ holomorphic on a neighborhood of $t = \infty$ which vanishes at $t = \infty$ of order $\frac{1}{2}\deg r - n$. Now $\int_{\mathfrak{o}}^{\mathfrak{s}} \frac{dx}{y} = t^{-n}\int_{\mathfrak{o}}^{\mathfrak{s}} \frac{dx_\infty}{y_\infty}$ so we can choose a branch of $\int_{\mathfrak{o}}^{\mathfrak{s}} \frac{dx}{y}$ of degree $-\frac{1}{2}\deg r$. Then apply Lemma 3 and we are done. $\square$

3 Algebraic de Rham cohomology

In this section we give a cohomological interpretation of Manin's map. I would like to give a brief description at the beginning, and then show the details. We use the following notation:

Notation 2. Using Notation 1, and

- Let $\tilde{E}$ be the minimal proper regular model of E.

- Denote by $\kappa : \tilde{E} \to \mathbf{P}^1_{\mathbb{C}}$ the elliptic fibration.
- Put $A = \mathbf{Spec}\,\mathbb{C}[t, \Delta^{-1}]$ and $E_\Delta = \kappa^{-1}A$.
- Let $\mathfrak{f}$ be the generic fiber of κ.
- Let $\mathfrak{s}$ be a section, and let $\mathfrak{o}$ be the 0-section.
- Put $H^1_{prim}(\widetilde{E}, \Omega^1_{\widetilde{E}}) = \{c \in H^1(\widetilde{E}, \Omega^1_{\widetilde{E}}); c.\mathfrak{f} = 0\}$, where $c.\mathfrak{f}$ denotes the intersection product.

Lemma 5. $\mathrm{Im}(H^1_{prim}(\widetilde{E}, \Omega^1_{\widetilde{E}}) \to H^1(E_\Delta, \Omega^1_{E_\Delta})) \subset \mathrm{Im}(H^1(E_\Delta, \kappa^*\Omega^1_A) \to H^1(E_\Delta, \Omega^1_{E_\Delta}))$.

Proof. For any $c \in H^1_{prim}(\widetilde{E}, \Omega^1_{\widetilde{E}})$, restrict c to any fiber $\mathfrak{f} \subset E_\Delta$ of κ, then by definition we have $0 = c|_{\mathfrak{f}} \in H^1(\mathfrak{f}, \Omega^1_{\mathfrak{f}})$. This means that c is 0 when viewed as an element of $H^0(A, R^1\kappa_*\Omega^1_{E_\Delta/A})$, and thus is 0 in $H^1(E_\Delta, \Omega^1_{E_\Delta/A})$. Then the exact sequence $H^1(E_\Delta, \kappa^*\Omega^1_A) \to H^1(E_\Delta, \Omega^1_{E_\Delta}) \to H^1(E_\Delta, \Omega^1_{E_\Delta/A})$ induced from $0 \to \kappa^*\Omega^1_A \to \Omega^1_{E_\Delta} \to \Omega^1_{E_\Delta/A} \to 0$, implies the lemma. □

For any $c \in H^1_{prim}(\widetilde{E}, \Omega^1_{\widetilde{E}})$, we take its correspondence $\hat{u} \in H^0(A, R^1\kappa_*\mathbb{C} \otimes \mathscr{O}_A)$ along the diagram:

$$\begin{array}{ccccc}
H^1_{prim}(\widetilde{E}, \Omega^1_{\widetilde{E}}) & & & & \\
\downarrow & & & & \\
H^1(E_\Delta, \Omega^1_{E_\Delta}) & & & & \\
\uparrow & & & & \\
H^1(E_\Delta, \kappa^*\Omega^1_A) & \xrightarrow{\cong} & H^0(A, R^1\kappa_*\kappa^*\Omega^1_A) & \longrightarrow & H^0(A, R^1\kappa_*\mathbb{C} \otimes \Omega^1_A) \\
 & & & & \| \\
 & & & & H^0(A, R^1\kappa_*\mathbb{C} \otimes \mathscr{O}_A) \cdot dt
\end{array}$$

and define a map $\tilde{\mu} : H^1_{prim}(\widetilde{E}, \Omega^1_{\widetilde{E}}) \to \mathbb{C}(t)$ by

$$\tilde{\mu}(c) = P_2(\hat{u} \smile \partial\omega) + (P_2\frac{\partial}{\partial t} + P_1)(\hat{u} \smile \omega)$$

where

$$\smile : H^0(A, R^1\kappa_*\mathbb{C} \otimes \mathscr{O}_A) \times H^0(A, R^1\kappa_*\mathbb{C} \otimes \mathscr{O}_A) \to H^0(A, R^2\kappa_*\mathbb{C} \otimes \mathscr{O}_A) \cong H^0(A, \mathscr{O}_A)$$

is the cup product and

$$\partial : H^0(A, R^1\kappa_*\mathbb{C} \otimes \mathscr{O}_A) \to H^0(A, R^1\kappa_*\mathbb{C} \otimes \mathscr{O}_A)$$

is the Gauss-Manin connection. This map is rather straightforward if we represent it using the algebraic de Rham cohomology. We will prove that Manin's map $\mu(\mathfrak{s})$ coincides with $\tilde{\mu}(c(\mathscr{O}_{\tilde{E}}(\mathfrak{s} - \mathfrak{o})))$ where $c(\mathscr{O}_{\tilde{E}}(\mathfrak{s} - \mathfrak{o}))$ is the Chern class of $\mathscr{O}_{\tilde{E}}(\mathfrak{s} - \mathfrak{o})$.

Now to show the details, we begin with a review on the algebraic de Rham cohomology. Let X be a separated scheme, and let $\mathfrak{U} = (U_i)_{i \in I}$ be an open affine covering of X, where I is a finite set endowed with a well-ordering. For any $i_0 < \ldots < i_p \in I$, denote by $U_{i_0\ldots i_p}$ the intersection $U_{i_0} \cap \ldots \cap U_{i_p}$, for a coherent sheaf $\mathscr{F}$ on X, put

$$C^p(\mathfrak{U},\mathscr{F}) = \bigoplus_{i_0<\ldots<i_p\in I} \mathscr{F}(U_{i_0\ldots i_p})$$

and define a coboundary map $\delta : C^p \to C^{p+1}$ by setting

$$(\delta\alpha)_{i_0\ldots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0\ldots\widehat{i_k}\ldots i_{p+1}}|_{U_{i_0\cdots i_{p+1}}}$$

Then $C^{\cdot}(\mathfrak{U},\mathscr{F})$ forms a complex whose cohomology is called the *Čech cohomology* and denoted by $H^{\cdot}(X,\mathscr{F})$. (cf. [7])

Now let X be a nonsingular variety over $\mathbb{C}$. The *algebraic de Rham cohomology* is the total cohomology of the following bicomplex $\mathscr{A}(X)$:

$$C^{\cdot}(X,\Omega_X^0) \xrightarrow{d} C^{\cdot}(X,\Omega_X^1) \to \ldots \to C^{\cdot}(X,\Omega_X^i) \to \ldots$$

Here $C^{\cdot}(X,\Omega_X^i)$ is the Čech complex for the sheaf Ω_X^i, and $\Omega_X^0 = \mathscr{O}_X$, $\Omega_X^i = \bigwedge^i \Omega_{X/\mathbb{C}}$, d is the exterior differential map. It is known that the algebraic de Rham cohomology is isomorphic to the singular cohomology $H^{\cdot}(X,\mathbb{C})$ (cf. [6], also [10]). When X is projective, this fact can be easily shown by the GAGA principle and Hodge's decomposition of $H^{\cdot}(X,\mathbb{C})$, via a filtration $F^0\mathscr{A} \supset F^1\mathscr{A} \supset \ldots \supset F^i\mathscr{A} \supset \ldots$ of $\mathscr{A}$ called the *Hodge filtration* defined by

$$F^i\mathscr{A} = (C^{\cdot}(X,\Omega_X^i) \to C^{\cdot}(X,\Omega_X^{i+1}) \to \ldots).$$

In particular when X is projective, the spectral sequence associated to the Hodge filtration degenerates at the E^1-level.

For a smooth morphism $\kappa : X \to Y$ of nonsingular varieties over $\mathbb{C}$, fix an open affine covering $\mathfrak{V} = (V_j)_{j\in J}$ of Y and an open covering $\mathfrak{U} = (U_i)_{i\in I}$ of X such that for any $i \in I$ and $j \in J$, $U_i \cap \kappa^{-1}V_j$ is affine ($\mathfrak{U}$ itself needs not to be affine, it can be taken as $\mathfrak{U} = \{X\}$ when κ is an affine morphism, or as $U_i = \{X_i \neq 0\}$ when κ is projective and factors through $\mathbf{Proj}\,\mathscr{O}_Y[X_0,\ldots,X_r]$). Denote by $\mathfrak{U}_{j_0\cdots j_p}$ the open affine covering of $\kappa^{-1}V_{j_0\cdots j_p}$ which is the restriction of $\mathfrak{U}$ on $\kappa^{-1}V_{j_0\cdots j_p}$, note that the total cohomology of the following bicomplex $\mathscr{C}^{\cdot}(X,\mathscr{F})$ also gives the Čech cohomology $H^{\cdot}(X,\mathscr{F})$:

$$\bigoplus_{j_0\in J} C^{\cdot}(\mathfrak{U}_{j_0},\mathscr{F}) \to \cdots \to \bigoplus_{j_0<\ldots<j_p\in J} C^{\cdot}(\mathfrak{U}_{j_0\cdots j_p},\mathscr{F}) \to \cdots$$

And if we regard the Čech complex $C^{\cdot}(X,\Omega_X^i)$ in the algebraic de Rham bicomplex $\mathscr{A}(X)$ as this bicomplex $\mathscr{C}^{\cdot}(X,\Omega_X^i)$ (thus $\mathscr{A}(X)$ becomes a "tricomplex"), we will have two filtrations of $\mathscr{A}(X)$, one is induced from a filtration of the Čech bicomplex $\mathscr{C}^{\cdot}(X,\star)$:

$$G^k\mathscr{C}^{\cdot}(X,\star) = (\bigoplus_{j_0<\ldots<j_k\in J} C^{\cdot}(\mathfrak{U}_{j_0\cdots j_k},\star) \to \bigoplus_{j_0<\ldots<j_{k+1}\in J} C^{\cdot}(\mathfrak{U}_{j_0\cdots j_{k+1}},\star) \to \cdots)$$

and the other one is induced from a filtration of the sheaf Ω_X^i:

$$L^j\Omega_X^i = \mathrm{Im}(\kappa^*\Omega_Y^j \otimes \Omega_X^{i-j} \to \Omega_X^j).$$

Then the filtration

$$T^i\mathscr{A} = \sum_{k+l=i} G^k\mathscr{A} \cap L^l\mathscr{A}$$

induces a spectral sequence whose E^1 and E^2 terms are

$$E^1 = \bigoplus_{i,k,l} \bigoplus_{j_0<\ldots<j_k\in J} \Gamma(V_{j_0\cdots j_k}, \Omega_Y^l \otimes R^i\kappa_*\mathbb{C}), E^2 = \bigoplus_{i,j} H^j(Y, R^i\kappa_*\mathbb{C})$$

This is the Leray-Hirsch spectral sequence associated to the morphism $\kappa : X \to Y$. When Y is affine and $\mathfrak{V} = \{Y\}$, the differential $d^1 : E^1 \to E^1$ can be viewed as the Gauss-Manin connection. (cf. [11])

Now let X be a nonsingular variety over $\mathbb{C}$, $\mathscr{L}$ an invertible sheaf over X, and let $\kappa : S \to X$ be the geometric $\mathbb{C}^\times$-bundle associated to $\mathscr{L}$. The Chern class $c(\mathscr{L})$ is by definition the image of $1 \in H^0(X, R^1\kappa_*\mathbb{Z})$ under the map $d^2 : H^0(X, R^1\kappa_*\mathbb{Z}) \to H^2(X, R^0\kappa_*\mathbb{Z})$, where d^2 is the differential $d^2 : E^2 \to E^2$ in the Leray-Hirsch spectral sequence associated to $\kappa : S \to X$. Thus the Chern class can be calculated as an element of the algebraic de Rham cohomology, using our description on the Leray-Hirsch spectral sequence as above. When $\mathscr{L}$ is given by an open affine covering $\mathfrak{V} = (V_i)_{i\in I}$ of X and the transformation functions $f_{ij} \in \Gamma(V_{ij}, \mathscr{O}_X^\times)$, it can be verified that $c(\mathscr{L})$ is defined by an element in $C^1(\mathfrak{V}, \Omega_X^1)$ as $c(\mathscr{L})|_{V_{ij}} = \frac{1}{2\pi\mathbf{i}}(df_{ij}/f_{ij})$. When we have a divisor D on X, it is easy to construct the invertible sheaf $\mathscr{O}_X(D)$ associated to D, then the Chern class $c(\mathscr{O}_X(D))$ is the cohomology class corresponding to D under the Poincaré duality.

I would like to rephrase the above general principles to fit the situation we are now considering. We will focus on $F^1Z_{DR}^2(E_\Delta)$ and $F^1H_{DR}^2(\tilde{E})$, where $Z_{DR}^2(\cdot)$ denotes the set of dimension 2 cocycles in the algebraic de Rham bicomplex, $H_{DR}^2(\cdot)$ the dimension 2 algebraic de Rham cohomology group, and $F^0 \supset F^1 \supset F^2$ is the Hodge filtration. For an affine open covering $\mathfrak{U}$ of E_Δ, we write down the bicomplex here:

$$\begin{tikzcd}
C^2(\mathfrak{U}, \mathscr{O}_{E_\Delta}) \arrow[r] & \cdot & \\
C^1(\mathfrak{U}, \mathscr{O}_{E_\Delta}) \arrow[u] \arrow[r, "d"] & C^1(\mathfrak{U}, \Omega_{E_\Delta}^1) \arrow[u, "\delta"] \arrow[r, "d"] & \cdot \\
& C^0(\mathfrak{U}, \Omega_{E_\Delta}^1) \arrow[u, "\delta"] \arrow[r, "d"] & C^0(\mathfrak{U}, \Omega_{E_\Delta}^2) \arrow[u]
\end{tikzcd}$$

Notation 3. Suffix like a_j^i denotes a Čech i-cochain of differential j-forms.

Thus an element in $F^1Z_{DR}^2(E_\Delta)$ is $\mu_1^1 + \mu_2^0 \in C^1(\mathfrak{U}, \Omega_{E_\Delta}^1) \oplus C^0(\mathfrak{U}, \Omega_{E_\Delta}^2)$ such that $\delta\mu_1^1 = 0$ and $d\mu_1^1 = \delta\mu_2^0$ (and, of course trivially, $d\mu_2^0 = 0$). We define an equivalence relation in $F^1Z_{DR}^2(E_\Delta)$:

Definition 1. For $\mu, \nu \in F^1 Z^2_{DR}(E_\Delta)$, we write $\mu \simeq \nu$ if there exists an element $\sigma^0_1 \in C^0(\mathfrak{U}, \Omega^1_{E_\Delta})$ such that $\mu - \nu = (\delta + d)\sigma^0_1$.

Recall that for nonsingular *projective* varieties, the spectral sequence associated to the Hodge filtration degenerates at the E^1-level, so it will be the same as the algebraic de Rham cohomologue relation if we define a '$\simeq$' relation there. However for the non-proper case E_Δ, the equivalence '$\simeq$' is a relation stronger than algebraic de Rham cohomologue. It will turn out to be important for us to rule out the influence from $C^1(\mathfrak{U}, \mathscr{O}_{E_\Delta})$.

The Gauss-Manin connection $D : \Gamma(A, \mathscr{O}_A \otimes R^1\kappa_*\mathbb{C}) \to \Gamma(A, \Omega^1_A \otimes R^1\kappa_*\mathbb{C})$ is described as follows: an element $\mu \in \Gamma(A, \mathscr{O}_A \otimes R^1\kappa_*\mathbb{C})$ can be represented by $\mu^1_0 + \mu^0_1 \in C^1(\mathfrak{U}, \mathscr{O}_{E_\Delta}) \oplus C^0(\mathfrak{U}, \Omega^1_{E_\Delta/A})$ such that $\delta\mu^1_0 = 0$ and $d\mu^1_0 = \delta\mu^0_1$, where $\Omega_{E_\Delta/A}$ denotes the sheaf of relative differential forms, and d is the relative differential. Thus, if we lift μ^0_1 to a $\tilde{\mu}^0_1 \in C^0(\mathfrak{U}, \Omega^1_{E_\Delta})$, the condition is to say that $\delta\mu^1_0 = 0$ and $d\mu^1_0 - \delta\tilde{\mu}^0_1 \in C^1(\mathfrak{U}, \kappa^*\Omega^1_A)$, here d denotes the total exterior differential. And of course $\kappa^*\Omega^1_A \otimes \Omega_{E_\Delta/A} = \Omega^2_{E_\Delta}$, so we define $D\mu$ to be $(-d\mu^1_0 + \delta\tilde{\mu}^0_1) + d\tilde{\mu}^0_1$, here d are all total differentials. $D\mu$ viewed as an element in $\Gamma(A, \Omega^1_A \otimes R^1\kappa_*\mathbb{C})$ does not depend on the choice of $\tilde{\mu}^0_1$. Here $\Gamma(A, \Omega^1_A \otimes R^1\kappa_*\mathbb{C})$ is regarded as the quotient of $C^1(\mathfrak{U}, \kappa^*\Omega^1_A) \oplus C^0(\mathfrak{U}, \kappa^*\Omega^1_A \otimes \Omega^1_{E_\Delta/A})$ by $(\delta + d)C^0(\mathfrak{U}, \kappa^*\Omega^1_A)$.

An element $u \in \Gamma(A, \Omega^1_A \otimes R^1\kappa_*\mathbb{C})$ can be written as $dt \wedge \hat{u}$, where $\hat{u} \in \Gamma(A, \mathscr{O}_A \otimes R^1\kappa_*\mathbb{C})$. Now if $D\mu = dt \wedge \hat{u}$, we will denote $\hat{u}$ by $D_{\frac{\partial}{\partial t}}\mu$. This map $D_{\frac{\partial}{\partial t}} : \Gamma(A, \mathscr{O}_A \otimes R^1\kappa_*\mathbb{C}) \to \Gamma(A, \mathscr{O}_A \otimes R^1\kappa_*\mathbb{C})$ is the Gauss-Manin connection, in the sense of §2, with respect to the derivation $\frac{\partial}{\partial t}$.

Take an affine open covering $\mathfrak{V} = (V_i)_{i \in I}$ of $\tilde{E}$ such that for each V_i we have a rational function $f_i \in K(\tilde{E})$ with $\mathrm{Zero}(f_i) \cap V_i = \mathfrak{o} \cap V_i$ and $\mathrm{Pole}(f_i) \cap V_i = \mathfrak{s} \cap V_i$. Then the Chern class $c^1_1 = c(\mathscr{O}_{\tilde{E}}(\mathfrak{s} - \mathfrak{o}))$ can be represented as an algebraic de Rham cohomology class by $(c^1_1)|_{V_{ij}} = \frac{1}{2\pi\mathbf{i}} d\log(f_j/f_i)$. Now Lemma 5 says that there exists a $\nu^0_1 \in C^0(\mathfrak{V}, \Omega^1_{E_\Delta})$ such that $c^1_1 - \delta\nu^0_1 \in C^1(\mathfrak{V}, \kappa^*\Omega^1_A)$. We put $u = c^1_1 - (\delta + d)\nu^0_1$. Then $u \simeq c^1_1$ and u can be viewed as an element in $\Gamma(A, \Omega^1_A \otimes R^1\kappa_*\mathbb{C})$. The element $\hat{u} \in \Gamma(A, \mathscr{O}_A \otimes R^1\kappa_*\mathbb{C})$ where $u = dt \wedge \hat{u}$ is what we will use to describe Manin's map $\mu(\mathfrak{s})$.

We first want to find a ν such that $D\nu = u$. This cannot be done algebraically, thus we do the following construction:

Definition 2. Denote by A_{an} the associated complex analytic space of A. For a point $\tau \in A_{an}$, take a small neighborhood N_τ of τ in the analytic topology. Put $F = \kappa^{-1}N_\tau$, and let $\mathfrak{W} = (W_j)_{j \in J}$ be an open covering of F which is a refinement of the restriction of $\mathfrak{V}$ on F. Thus for any W_j there is a $V_i \supset W_j$, and we assign to W_j the meromorphic function g_j, which is (the restriction of) the rational function f_i assigned to V_i, in the definition of c^1_1. We define $\nu^1_0 \in C^1(\mathfrak{W}, \mathscr{O}^{an}_F)$ to be an analytic Čech cochain where $(\nu^1_0)|_{W_{jk}} = \frac{-1}{2\pi\mathbf{i}} \log(g_k/g_j)$.

Lemma 6. *Taking N_τ and $\mathfrak{W}$ to be sufficiently fine we can fix a branch of* $\log(g_k/g_j)$ *on each W_{jk} to make ν^1_0 a Čech cocycle.*

Proof. Put $\rho_0^1 \in C^1(\mathfrak{W}, \mathscr{O}_F^{an\times})$ to be $(\rho_0^1)|_{W_{jk}} = g_k/g_j$. The Chern class of the line bundle $\mathscr{O}_{E_\Delta}(\mathfrak{s}-\mathfrak{o})$ restricted to F can also be viewed as the image of ρ_0^1 under the connecting map $H^1(F, \mathscr{O}_F^{an\times}) \to H^2(F, \mathbb{Z})$ induced from the exact sequence

$$0 \to \mathbb{Z} \to \mathscr{O}_F^{an} \xrightarrow{e^{2\pi\mathbf{i}\cdot}} \mathscr{O}_F^{an\times} \to 0.$$

Now the line bundle is trivial for sufficiently small N_τ so the image of ρ_0^1 is 0. Then the lemma follows from the construction of the connecting map. □

Now we can put together ν_0^1 and ν_1^0 to define $\nu = \nu_0^1 - \nu_1^0$, and ν can be viewed as an element in $\Gamma(N_\tau, \mathscr{O}_{N_\tau}^{an} \otimes R^1\kappa_*\mathbb{C})$. The Gauss-Manin connection can be defined parallelly in the analytic case, namely $D\nu = (-d\nu_0^1 - \delta\nu_1^0) - d\nu_1^0 = c_1^1 - (\delta + d)\nu_1^0 = u$. The construction of ν is of course local for each $\tau \in A$, however $D\nu$ glues to a global algebraic cocycle u.

Let $\omega \in \Gamma(A, \kappa_*\Omega^1_{E_\Delta/A})$ be a relative 1-form of the first kind. View ω as an element in $\Gamma(A, \mathscr{O}_A \otimes R^1\kappa_*\mathbb{C})$ and consider the cup product $\nu \smile \omega \in \Gamma(N_\tau, \mathscr{O}_{N_\tau}^{an} \otimes R^2\kappa_*\mathbb{C})$. Since the fibers of κ are oriented manifolds of real dimension 2 we have a natural sheaf isomorphism $R^2\kappa_*\mathbb{C} \cong \mathbb{C}$, hence $\nu \smile \omega$ can be viewed as an analytic function on N_τ.

Lemma 7. *We have $\nu \smile \omega = \int_{\mathfrak{o}}^{\mathfrak{s}} \omega$. The ambiguity of the choice of an integral path comes from the choice of branches of* $\log(g_k/g_j)$ *in Definition* 2.

Proof. We argue on each fiber $\mathfrak{f} \subset F$ of κ. Denote the open covering $\mathfrak{W}$ restricted to $\mathfrak{f}$ also by $\mathfrak{W}$. Consider the bicomplex

$$\begin{array}{ccc} C^1(\mathfrak{W}, \mathscr{D}_{\mathfrak{f}}^1) & \xrightarrow{d} & \cdot \\ \uparrow{\scriptstyle\delta} & & \uparrow \\ C^0(\mathfrak{W}, \mathscr{D}_{\mathfrak{f}}^1) & \xrightarrow{d} & C^0(\mathfrak{W}, \mathscr{D}_{\mathfrak{f}}^2) \end{array}$$

where $\mathscr{D}_{\mathfrak{f}}^i$ are sheaves of currents of degree i. The sheaves $\mathscr{D}_{\mathfrak{f}}^i$ are fine, and it is known that the de Rham complex with currents gives the de Rham cohomology, so the above bicomplex also does. We can regard $\nu \smile \omega$, where $(\nu \smile \omega)|_{W_{jk}} = \frac{-1}{2\pi\mathbf{i}}\log(g_k/g_j)\omega$, as an element in $C^1(\mathfrak{W}, \mathscr{D}_{\mathfrak{f}}^1)$. Fix a cut line γ on $\mathfrak{f}$ from $\mathfrak{o} \cap \mathfrak{f}$ to $\mathfrak{s} \cap \mathfrak{f}$. Refine $\mathfrak{W}$ if necessary, we can find an element $z \in C^1(\mathfrak{W}, \mathbb{Z})$ such that $\delta z = 0$ and $(\nu \smile \omega) - z\omega = \delta\sigma$, where $\sigma \in C^0(\mathfrak{W}, \mathscr{D}_{\mathfrak{f}}^1)$ and $(\sigma)|_{W_j} = \frac{1}{2\pi\mathbf{i}}\log(g_j)\omega$, the branch of $\log(g_j)$ is taken to be such that the only discontinuities of $\log(g_j)$ are on the cut line γ, and the differences of values of $\log(g_j)$ between the two sides of γ are just $2\pi\mathbf{i}$. Then $\nu \smile \omega$ is cohomologous to $z\omega - d\sigma$, where z can be viewed as an element in $H^1(\mathfrak{f}, \mathbb{Z})$ which comes from the choice of an integral path or a branch of $\log(g_k/g_j)$, and $d\sigma$ is the differential of σ in the sense of a current. Take the fundamental class $[\mathfrak{f}]$ of $\mathfrak{f}$ we have $-d\sigma([\mathfrak{f}]) = -\int_{[\mathfrak{f}]} d\sigma = -\int_{\partial(\mathfrak{f}\setminus\gamma)} \sigma = \int_\gamma \omega$, which proves the statement. □

Put $\partial = \frac{\partial}{\partial t}$, $\omega = \frac{dx}{y}$ and let Manin's map μ be defined by the Picard-Fuchs equation $P_2\partial\partial + P_1\partial + P_0$ with respect to ω. Then using Lemma 7 we can apart ∂ to each side of the cup product, while by definition we have $(P_2 D_{\frac{\partial}{\partial t}} D_{\frac{\partial}{\partial t}} + P_1 D_{\frac{\partial}{\partial t}} + P_0)\omega = 0$, so

$$\begin{aligned}\mu(\mathfrak{s}) &= (P_2 D_{\frac{\partial}{\partial t}} D_{\frac{\partial}{\partial t}} + P_1 D_{\frac{\partial}{\partial t}} + P_0)(\nu \smile \omega) \\ &= P_2(D_{\frac{\partial}{\partial t}} \nu \smile D_{\frac{\partial}{\partial t}} \omega) + (P_2 \frac{\partial}{\partial t} + P_1)(D_{\frac{\partial}{\partial t}} \nu \smile \omega) \\ &= P_2(\hat{u} \smile D_{\frac{\partial}{\partial t}} \omega) + (P_2 \frac{\partial}{\partial t} + P_1)(\hat{u} \smile \omega)\end{aligned} \tag{10}$$

Note that, the cup product $\nu \smile \omega$ does *not* depend on the choice of ν_1^0, thus is determined only by c_1^1. So we can choose *any* u such that $u \simeq c_1^1$ and $u \in \Gamma(A, \Omega_A^1 \otimes R^1 \kappa_* \mathbb{C})$, in the calculation of $\mu(\mathfrak{s})$. This is the merit to use relative 1-forms of the first kind; also the reason for us to define the relation '$\simeq$'.

Definition 3. Now we can define a $\mathbb{C}$-linear map $\tilde{\mu} : H^1_{prim}(\widetilde{E}, \Omega^1_{\widetilde{E}}) \oplus H^0(\widetilde{E}, \Omega^2_{\widetilde{E}}) \to \mathbb{C}(t)$ as follows: for any $c \in H^1_{prim}(\widetilde{E}, \Omega^1_{\widetilde{E}}) \oplus H^0(\widetilde{E}, \Omega^2_{\widetilde{E}})$, by Lemma 5 we can find a $\hat{u} \in \Gamma(A, \mathscr{O}_A \otimes R^1 \kappa_* \mathbb{C})$ such that $c|_{E_\Delta} \simeq dt \wedge \hat{u}$. Then we define

$$\tilde{\mu}(c) = P_2(\hat{u} \smile D_{\frac{\partial}{\partial t}} \omega) + (P_2 \frac{\partial}{\partial t} + P_1)(\hat{u} \smile \omega).$$

This generalizes Manin's map μ, and provides a cohomological interpretation.

Recall that $h^{1.1}(\tilde{E}) = 10n$, $h^{2.0}(\tilde{E}) = n - 1$. Let $W \subset H^{1.1}(\tilde{E})$ be the subspace generated by "trivial algebraic cycles", *i.e.* the 0-section, generic fiber, and all fiber components which do not intersect the 0-section. We have an orthogonal decomposition $H^{1.1}(\tilde{E}) = W \oplus W^\perp$. The Mordell-Weil lattice of $\tilde{E}$ is a lattice of an $\mathbb{R}$-subspace of $W^\perp_{\mathbb{R}} = W^\perp \cap H^2(\tilde{E}, \mathbb{R})$, and $\dim_{\mathbb{C}} W^\perp = 10n - 2 - \sum_\nu (m_\nu - 1)$ where m_ν is the number of fiber components and ν runs over all (non-irreducible singular) fibers of $\tilde{E}$. (cf. [19])

$W \cap H^1_{prim}(\widetilde{E}, \Omega^1_{\widetilde{E}})$ is generated by fiber components while $W^\perp \subset H^1_{prim}(\widetilde{E}, \Omega^1_{\widetilde{E}})$. We have $\tilde{\mu}(W \cap H^1_{prim}(\widetilde{E}, \Omega^1_{\widetilde{E}})) = 0$, because the line bundle (and thus its Chern class) associated to a fiber component is trivial when restricted to E_Δ. So $\tilde{\mu}$ is essentially a map from $W^\perp \oplus H^{2.0}(\widetilde{E})$ to $\mathbb{C}(t)$.

4 Proof of the main theorem

By an explicit calculation of the algebraic de Rham cohomology, we can get the image of $\tilde{\mu}$. The cohomology calculation is summarized in the following key lemma which will be proved in §7:

Lemma 8 (Key Lemma)**.** *Regard $\omega = \frac{dx}{y}$ as an element in $\Gamma(A, \mathscr{O}_A \otimes R^1 \kappa_* \mathbb{C})$. There exists an $\eta \in \Gamma(A, \mathscr{O}_A \otimes R^1 \kappa_* \mathbb{C})$ such that:*

1. $\eta \smile \omega = 4\pi \mathbf{i}$.
2. *The Gauss-Manin connection is*

$$D_{\frac{\partial}{\partial t}} \begin{pmatrix} \eta \\ \omega \end{pmatrix} = \frac{1}{12\Delta} \begin{pmatrix} \Delta' & -a\Lambda \\ \Lambda & -\Delta' \end{pmatrix} \begin{pmatrix} \eta \\ \omega \end{pmatrix}.$$

3. *For any $p,q \in \mathbb{C}[t]$ and $p_\infty, q_\infty \in \mathbb{C}[t^{-1}]$, there exists $u \in F^1H^2_{DR}(\tilde{E})$ such that $u|_{E_\Delta} \simeq dt \wedge \upsilon$ where*

$$\upsilon = (-\frac{1}{2}\frac{\Delta'}{\Delta}p - 6p' + q)\eta + (\frac{1}{2}\frac{a\Lambda}{\Delta}p + \lfloor \frac{12}{\Lambda^2}\rho(p - t^{-n}p_\infty, q - t^{-n-2}q_\infty)\rfloor)\omega.$$

In this section, we use Lemma 8 to write down elements in $\mathrm{Im}\tilde{\mu}$, then by a dimension counting, we prove that $\mathrm{Im}\tilde{\mu} \subset \mathbb{C}[t]$ and $\tilde{\mu} : W^\perp \oplus H^{2,0}(\tilde{E}) \to \mathbb{C}[t]$ is injective. This together with Proposition 6 will imply the main theorem.

First, for a u as in 3. of Lemma 8, calculating by Definition 3 we get

$$\frac{1}{4\pi\mathbf{i}}\tilde{\mu}(u) = \lfloor \frac{1}{\Lambda^2}\rho(t^{-n}p_\infty, t^{-n-2}q_\infty)\rfloor \Lambda^2 + \rho(p,q) \text{ MOD } \Lambda^2. \tag{11}$$

Then we count the dimension of the vector space consisting of these elements.

Lemma 9. *We have $\dim_{\mathbb{C}} B = \deg\Lambda - \sum_{\nu\neq\infty}(m_\nu - 1)$, where m_ν is the number of fiber components and ν runs over all but the ∞-fiber of $\tilde{E}$.*

Proof. This is proved by a local calculation. Put $R = \{\rho(p,q); p,q \in \mathbb{C}[t]\}$. For any $w \in \mathbb{C}$, we have a natural map $\pi_w : \mathbb{C}[t] \to \mathbb{C}[t]/((t-w)^{2\,\mathrm{ord}_w\Lambda})$. Then by the Chinese Remainder Theorem, $\dim_{\mathbb{C}} B = \sum_w \dim_{\mathbb{C}} \pi_w(R)$. Now fix a $w \in \mathbb{C}$, from Lemma 4 we see that $\mathrm{ord}_w(P_2p'' + P_1p' + P_0p)$ can be any integer $\geq \mathrm{ord}_w(\Delta\Lambda) - 2$ except for

1. $\mathrm{ord}_w\Lambda - 2$ and $2\,\mathrm{ord}_w\Lambda - 1$, if the w-fiber is nonsingular ($\mathrm{ord}_w\Delta = 0$)
2. $\mathrm{ord}_w(\Delta\Lambda) - 2$, if the w-fiber is of type I_m $(m \geq 1)$

On the other hand it is easy to check that $\mathrm{ord}_w(\Delta\Lambda q' - \Delta\Lambda' q + \frac{11}{12}\Delta'\Lambda q)$ can be any integer $\geq \mathrm{ord}_w(\Delta\Lambda) - 1$ except for $2\,\mathrm{ord}_w\Lambda - 1$ if the w-fiber is nonsingular (and $\mathrm{ord}_w\Delta = 0$). So

1. If the w-fiber is nonsingular, $\mathrm{ord}_w\rho(p,q)$ can be any integer $\geq \mathrm{ord}_w\Lambda - 1$ except for $2\,\mathrm{ord}_w\Lambda - 1$. So $\dim_{\mathbb{C}} \pi_w(R) = \mathrm{ord}_w\Lambda$.
2. If the w-fiber is of type I_m $(m \geq 1)$, $\mathrm{ord}_w\rho(p,q)$ can be any integer $\geq \mathrm{ord}_w(\Delta\Lambda) - 1$. In this case $\mathrm{ord}_w\Lambda = \mathrm{ord}_w\Delta - 1$ and the number of w-fiber components $m_w = \mathrm{ord}_w\Delta$, so indeed $\dim_{\mathbb{C}} \pi_w(R) = 0 = \mathrm{ord}_w\Lambda - (m_w - 1)$.
3. Otherwise, $\mathrm{ord}_w\rho(p,q)$ can be any integer $\geq \mathrm{ord}_w(\Delta\Lambda) - 2$. In this case $m_w = \mathrm{ord}_w\Delta - 1$, we see also $\dim_{\mathbb{C}} \pi_w(R) = \mathrm{ord}_w\Lambda - \mathrm{ord}_w\Delta + 2 = \mathrm{ord}_w\Lambda - (m_w - 1)$.

Now note that $\deg\Lambda = \sum_w \mathrm{ord}_w\Lambda$ so we are done. □

Lemma 10. *Put $\Gamma = \{\lfloor \frac{1}{\Lambda^2}\rho(t^{-n}p_\infty, t^{-n-2}q_\infty)\rfloor; p_\infty, q_\infty \in \mathbb{C}[t^{-1}]\}$. Then*

1. *If the ∞-fiber is nonsigular, then $\dim_{\mathbb{C}}\Gamma = \deg\Delta - \deg\Lambda - n - 2$.*
2. *If the ∞-fiber is of type I_m $(m \geq 1)$, then $\dim_{\mathbb{C}}\Gamma = \deg\Delta - \deg\Lambda - n - 1$.*
3. *Otherwise $\dim_{\mathbb{C}}\Gamma = \deg\Delta - \deg\Lambda - n$.*

Proof. Similar to the proof of Lemma 9, using Lemma 3 instead of Lemma 4. □

Lemma 11. *We have* $\mathrm{Im}\tilde{\mu} \subset \mathbb{C}[t]$ *and* $\tilde{\mu} : W^{\perp} \oplus H^{2.0}(\tilde{E}) \to \mathbb{C}[t]$ *is injective.*

Proof. By Lemma 9, Lemma 10 and (11), we have already produced a sub vector space of $\mathrm{Im}\tilde{\mu}$ consisting of polynomials and whose dimension is $\dim_{\mathbb{C}} B + \dim_{\mathbb{C}} \Gamma = 11n-3-\sum_v(m_v-1)$. Now since $\dim_{\mathbb{C}}(W^{\perp} \oplus H^{2.0}(\tilde{E})) = 11n-3-\sum_v(m_v-1)$, all elements in $\mathrm{Im}\tilde{\mu}$ are already obtained and $\tilde{\mu} : W^{\perp} \oplus H^{2.0}(\tilde{E}) \to \mathbb{C}[t]$ is injective. □

Proof of Theorem 1. For $\mathfrak{s} = (r,s) \in E(\mathbb{C}(t))$, by Proposition 6 we can obtain $\frac{1}{2}\deg r$ from $\deg \mu(\mathfrak{s})$, and by the arguments above we know that all the possible forms of $\mu(\mathfrak{s})$ is of the form in (11). So all the possiblilities of $\frac{1}{2}\deg r$ are known. To be more precisely,

1. Since $c \le 2\deg\Lambda - 1$ so $\deg(\Delta\Lambda) - c - 2 \ge \deg\Delta - \deg\Lambda - 1 \ge 2n-1$, then it does not matter for us to assume $\frac{1}{2}\deg r > n$. If $\deg\Delta = 12n$ we can further assume $\frac{1}{2}\deg r > 11n - \deg\Lambda - 1$. Anyway by Proposition 6 we have $\frac{1}{2}\deg r = \deg(\Delta\Lambda) - \deg\mu(\mathfrak{s}) - 2$. From (11) and the definition of c, we get $\deg\mu(\mathfrak{s}) \ge c$.
2. We have $\tilde{\mu}(H^{2.0}(\tilde{E})) = \{l\Lambda^2; l \in \mathbb{C}[t], \deg l \le n-2\}$, and elements of $\tilde{\mu}(W^{\perp})$ can be written as the form $\gamma\Lambda^2 + \beta$, where $\gamma \in \Gamma$ and $\beta \in B$. Now
   (a) For $c \le j \le 2\deg\Lambda - 1$, we have
   $$\frac{1}{2}\deg r \ge \deg(\Delta\Lambda) - j - 2 \;\Rightarrow\; \deg\mu(\mathfrak{s}) \le j \;\Rightarrow\; \gamma = 0, \deg\beta \le j.$$
   (b) For $2\deg\Lambda + n - 2 \le k$,
   $$\frac{1}{2}\deg r \ge \deg(\Delta\Lambda) - k - 2 \;\Rightarrow\; \deg\mu(\mathfrak{s}) \le k \;\Rightarrow\; \deg(\gamma\Lambda^2) \le k.$$

Since the Mordell-Weil lattice is a lattice of an $\mathbb{R}$-subspace of $W_{\mathbb{R}}^{\perp}$, and $\tilde{\mu} : W^{\perp} \to \mathbb{C}[t]$ is injective, so the dimension estimates for subspaces of $\mathbb{C}[t]$ implies rank estimates for sublattices of the Mordell-Weil lattice. □

5 Calculation of Čech cohomology

In this section I explain the method I used to calculate the Čech cohomology of a coherent sheaf $\mathscr{F}$.

Let A be a noetherian ring, $X \subset \mathbf{P}_A^r = \mathbf{Proj}A[X_0, \ldots, X_r]$ a projective scheme, and let $\mathfrak{U} = (U_i)_{0 \le i \le r}$ be the canonical open covering such that $U_i = \{X_i \ne 0\}$. To calculate the Čech cohomology under this setup, we take a free resolution $\mathscr{F} \leftarrow \mathfrak{F}.$ in the form:

$$0 \leftarrow \mathscr{F} \leftarrow \bigoplus_i \mathscr{O}_{\mathbf{P}_A^r}(-n_i) \leftarrow \cdots \leftarrow \bigoplus_j \mathscr{O}_{\mathbf{P}_A^r}(-n_j) \leftarrow 0$$

(that it can be taken with length less than r is a consequence of Hilbert's syzygy theorem) and consider the bicomplex:

$$\begin{array}{ccccccc}
 & & H^r(\mathbf{P}^r_A,\mathfrak{F}_0) & \longleftarrow & \dots & \longleftarrow & H^r(\mathbf{P}^r_A,\mathfrak{F}_r) \\
 & & \uparrow & & & & \uparrow \\
C^r(\mathfrak{U},\mathscr{F}) & \longleftarrow & C^r(\mathfrak{U},\mathfrak{F}_0) & \longleftarrow & \dots & \longleftarrow & C^r(\mathfrak{U},\mathfrak{F}_r) \\
\uparrow & & \uparrow & & & & \uparrow \\
\vdots & & \vdots & & & & \vdots \\
\uparrow & & \uparrow & & & & \uparrow \\
C^0(\mathfrak{U},\mathscr{F}) & \longleftarrow & C^0(\mathfrak{U},\mathfrak{F}_0) & \longleftarrow & \dots & \longleftarrow & C^0(\mathfrak{U},\mathfrak{F}_r)
\end{array}$$

It is well-known that if $n \geq 1$ we have $H^i(\mathbf{P}^r_A, \mathscr{O}_{\mathbf{P}^r_A}(-n)) = 0$ for $0 \leq i \leq r-1$, and $\{X_0^{l_0} \cdots X_r^{l_r} | l_i \leq -1, \sum l_i = -n\}$ (viewed as local sections on the open set $U_{0\dots r}$) is a basis of $H^r(\mathbf{P}^r_A, \mathscr{O}_{\mathbf{P}^r_A}(-n))$, so by the routine argument on a bicomplex we get an isomorphism $H^i(X,\mathscr{F}) \cong h_{r-i}(H^r(\mathbf{P}^r_A, \mathfrak{F}.))$, however to practically use this to calculate $H^i(X,\mathscr{F})$, *i.e.* to carry out the diagram chasing of the bicomplex, some remarks should be made:

- It is a standard task to calculate the free resolution of a coherent sheaf on a projective scheme, using gröbner basis. (cf.[1], and also [3]) For the use in this paper, free resolutions are explicitly given, however calculation with gröbner basis is still necessary to "pull back the row", *i.e.* for a local section s which satisfy $\phi(s) = 0$, to find a local section t such that $s = \phi(t)$. Here ϕ denotes the boundary map of the free resolution.
- In order to "pull back the column", *i.e.* for a $(p+1)$-Čech coboundary β of the sheaf $\mathscr{O}_{\mathbf{P}^r_A}(-n)$, to find an $\alpha \in C^p(\mathfrak{U}, \mathscr{O}_{\mathbf{P}^r_A}(-n))$ such that $\delta\alpha = \beta$, the following chain homotopy map $\Phi : C^{p+1} \to C^p$ can be used:

  $$(\Phi\beta)_{0\cdots k\, i_{k+1}\cdots i_p} = (-1)^k (\beta_{0\cdots k(k+1)\, i_{k+1}\cdots i_p})_{k+1}$$

  Here $k+1 < i_{k+1} < \cdots < i_p \leq r$, and if we write $\beta_{0\cdots k(k+1)\, i_{k+1}\cdots i_p}$ in the form $\sum c_{l_0\cdots l_r} X_0^{l_0} \cdots X_r^{l_r}$ (where $c_{l_0\cdots l_r} \in A$), then $(\beta_{0\cdots k(k+1)\, i_{k+1}\cdots i_p})_{k+1}$ means to take the sum of such $(c_{l_0\cdots l_r} X_0^{l_0} \cdots X_r^{l_r})$s that $l_j \leq -1$ for all $0 \leq j \leq k$ and $l_{k+1} \geq 0$. Note that, restricted to every fixed $X_0^{l_0} \cdots X_r^{l_r}$, denoting the set $\{i | l_i \geq 0\}$ by $|l|$, this chain homotopy map Φ is just the chain homotopy map for the A-coefficient complex of a simplex whose vertexes are labeled by the set $|l|$. The necessity to use this Φ is the reason why we should take the canonical open covering $\mathfrak{U}$ in the calculation.
- An experimental implementation using SINGULAR can be found at [21].

To deal with other open affine coverings, say, $\mathfrak{V} = (V_j)_{j\in J}$, define

$$C^{p.q} = \bigoplus_{\substack{0\le i_0<\ldots<i_p\le r \\ j_0<\ldots<j_q\in J}} \mathscr{F}(U_{i_0\ldots i_p} \cap V_{j_0\ldots j_q})$$

then the bicomplex

$$\begin{tikzcd}
C^r(\mathfrak{U},\mathscr{F}) \arrow[r] & C^{r.0} \arrow[r] & \ldots \arrow[r] & C^{r.s} \\
\vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
C^0(\mathfrak{U},\mathscr{F}) \arrow[r] \arrow[u] & C^{0.0} \arrow[r] \arrow[u] & \ldots \arrow[r] & C^{0.s} \arrow[u] \\
& C^0(\mathfrak{V},\mathscr{F}) \arrow[r] \arrow[u] & \ldots \arrow[r] & C^s(\mathfrak{V},\mathscr{F}) \arrow[u]
\end{tikzcd}$$

will relate $C^{\cdot}(\mathfrak{V},\mathscr{F})$ with $C^{\cdot}(\mathfrak{U},\mathscr{F})$. To actually perform the diagram chasing, all we should know is a way to "pull back" the following type of exact sequences:

$$0 \to M \to \bigoplus_{r_0\in I} M_{r_0} \xrightarrow{\delta} \ldots \to \bigoplus_{r_0<\ldots<r_p\in I} M_{r_0\cdots r_p} \to \ldots$$

Here M is a finitely generated R-module where R is a finitely generated A-algebra. M_{r_0} *etc.* denote the localizations of M. $I \subset R$ is a finite set endowed with a well-ordering and the ideal generated by I is R. The pull-back can be done as follows. For a p-coboundary $\beta \in \bigoplus M_{r_0\cdots r_p}$, take m sufficiently large such that $r_i^m(\beta)_{r_0\cdots r_p} \in M_{r_0\cdots\widehat{r_i}\cdots r_p}$ for all $r_0 < \ldots < r_p \in I$ and all $0 \le i \le p$. Then take a "division of the unity" $1 = \sum c_{r_i} r_i^m$ and define $\alpha \in \bigoplus M_{r_0\cdots r_{p-1}}$ to be

$$\alpha = \sum_{r_0<\ldots<r_p\in I} \sum_{0\le i\le p} (-1)^i c_{r_i}(r_i^m(\beta)_{r_0\cdots r_p})$$

where $(r_i^m(\beta)_{r_0\cdots r_p})$ is viewed as an element of $M_{r_0\cdots\widehat{r_i}\cdots r_p}$. Thus we have $\delta\alpha = \beta$.

6 Free resolutions for hypersurfaces

Let A be a Cohen-Macaulay integral $\mathbb{C}$-algebra, put $\mathbf{P}_A^r = \mathbf{Proj} A[X_0,\ldots,X_r]$, and let $X \subset \mathbf{P}_A^r$ be a hypersurface defined by a homogeneous polynomial $f \in A[X_0,\ldots,X_r]$ of degree m. Assume that X is smooth over the generic point of $\mathbf{Spec} A$. In this section we will give an explicit free resolution of the sheaf Ω_X^i.

Lemma 12 (Koszul Complex)**.** *Let $x_0,\ldots,x_n \in B$ be a regular sequence for a ring B. Define a complex $K.$ by*

$$K_p := \bigoplus_{0\le i_0<\cdots<i_p\le n} B\mathbf{e}_{i_0\cdots i_p},\ d(\mathbf{e}_{i_0\cdots i_p}) := \sum_{k=0}^{p} (-1)^k x_{i_k} \mathbf{e}_{i_0\cdots\widehat{i_k}\cdots i_p}$$

Then $H_i(K.) = 0$ for $i \geq 0$ and $H_{-1}(K.) = B/(x_0,\ldots,x_n)$. If B is Cohen-Macaulay and $\dim B - \dim B/(x_0,\ldots,x_n) = n+1$, *then $x_0,\ldots,x_n$ is a regular sequence.*

Proof. cf. for example [15]. $\square$

Notation 4. Let $\widetilde{\Omega}$ and $\widetilde{\Omega}^i$ be the locally free sheaves on $\mathbf{P}^r_A$ defined by

$$\widetilde{\Omega} = \bigoplus_{k=0}^{r} \mathscr{O}_{\mathbf{P}^r_A}(-1)\, dX_k,\ \widetilde{\Omega}^i = \bigwedge^i \widetilde{\Omega}.$$

Here dX_k is a formal symbol.

Apply Lemma 12 to the ring $A[X_0,\ldots,X_r]$ with regular sequence $(X_0,\ldots,X_r)$, we get the following exact sequence:

$$0 \leftarrow \mathscr{O}_{\mathbf{P}^r_A} \stackrel{\iota_\theta}{\leftarrow} \widetilde{\Omega} \stackrel{\iota_\theta}{\leftarrow} \widetilde{\Omega}^2 \leftarrow \cdots \leftarrow \widetilde{\Omega}^{r+1} \leftarrow 0 \tag{12}$$

Here the differential map ι_θ can be viewed as the inner product with the formal vector field $\theta = X_0 \frac{\partial}{\partial X_0} + \cdots + X_r \frac{\partial}{\partial X_r}$ (and thus the notation).

Lemma 13. *Regard $\Omega^i_{\mathbf{P}^r_A/A}$ as a subsheaf of $\widetilde{\Omega}^i$, then the following gives a free resolution of $\Omega^i_{\mathbf{P}^r_A/A}$:*

$$0 \leftarrow \Omega^i_{\mathbf{P}^r_A/A} \stackrel{\iota_\theta}{\leftarrow} \widetilde{\Omega}^{i+1} \stackrel{\iota_\theta}{\leftarrow} \widetilde{\Omega}^{i+2} \leftarrow \cdots \leftarrow \widetilde{\Omega}^{r+1} \leftarrow 0$$

Proof. An i-form $\alpha \in \widetilde{\Omega}^i$ comes from $\Omega^i_{\mathbf{P}^r_A/A}$, if and only if $\iota_\theta \alpha = 0$ (elementary calculation, or cf.[5, Prop.2.2]). So it follows immediately from (12). $\square$

Denote the quotient field of A by K. Since X is smooth over the generic point of $\mathbf{Spec} A$, we have $\dim K[X_0,\ldots,X_r]/(\frac{\partial}{\partial X_0} f,\ldots,\frac{\partial}{\partial X_r} f) = 0$, so $(\frac{\partial}{\partial X_0} f,\ldots,\frac{\partial}{\partial X_r} f)$ is a regular sequence of $A[X_0,\ldots,X_r]$ assuming that A is Cohen-Macaulay. Apply Lemma 12 to this regular sequence we get that

$$\widetilde{\Omega}^{r+1} \stackrel{df\wedge\cdot}{\leftarrow} \widetilde{\Omega}^{r}(-m) \stackrel{df\wedge\cdot}{\leftarrow} \widetilde{\Omega}^{r-1}(-2m) \leftarrow \cdots \tag{13}$$

is an exact sequence. Here the differential map can be viewed as the wedge product with $df = \frac{\partial}{\partial X_0} f dX_0 + \cdots + \frac{\partial}{\partial X_r} f dX_r$.

Similarly since $K[\frac{X_0}{X_i},\ldots,\widehat{\frac{X_i}{X_i}},\ldots,\frac{X_r}{X_i}]/(\frac{f}{X_i^m},\frac{\frac{\partial}{\partial X_0} f}{X_i^{m-1}},\ldots,\widehat{\frac{\frac{\partial}{\partial X_i} f}{X_i^{m-1}}},\ldots,\frac{\frac{\partial}{\partial X_r} f}{X_i^{m-1}}) = 0$ for all $0 \leq i \leq r$, we conclude that $(\frac{\frac{\partial}{\partial X_0} f}{X_i^{m-1}},\ldots,\widehat{\frac{\frac{\partial}{\partial X_i} f}{X_i^{m-1}}},\ldots,\frac{\frac{\partial}{\partial X_r} f}{X_i^{m-1}})$ is a regular sequence of the ring $A[\frac{X_0}{X_i},\ldots,\widehat{\frac{X_i}{X_i}},\ldots,\frac{X_r}{X_i}]/(\frac{f}{X_i^m})$, apply Lemma 12 to these and glue all i, we get the following exact sequence:

$$\Omega^r_{\mathbf{P}^r_A/A} \otimes \mathscr{O}_X \stackrel{df\wedge\cdot}{\leftarrow} \Omega^{r-1}_{\mathbf{P}^r_A/A} \otimes \mathscr{O}_X(-m) \stackrel{df\wedge\cdot}{\leftarrow} \Omega^{r-2}_{\mathbf{P}^r_A/A} \otimes \mathscr{O}_X(-2m) \leftarrow \cdots \tag{14}$$

Here $\Omega^i_{\mathbf{P}^r_A/A}$ is regarded as a subsheaf of $\widetilde{\Omega}^i$. Note that if an i-form $\alpha \in \widetilde{\Omega}^i$ satisfies $\iota_\theta \alpha = 0$, we have $\iota_\theta(df \wedge \alpha) = mf\alpha \equiv 0 \mod (f)$, so the differential map in (14) is well-defined.

Lemma 14. *The following is a free resolution for $\Omega^i_{X/A}$:*

$$0 \leftarrow \Omega^i_{X/A} \xleftarrow{\iota_\theta} \widetilde{\Omega}^{i+1} \xleftarrow{\iota_\theta + df\wedge\cdot} \widetilde{\Omega}^{i+2} \oplus \widetilde{\Omega}^i(-m) \xleftarrow{\iota_\theta \oplus df\wedge\cdot} \widetilde{\Omega}^{i+3} \oplus \widetilde{\Omega}^{i-1}(-2m) \xleftarrow{\iota_\theta \oplus df\wedge\cdot} \cdots$$

Proof. $\Omega^i_{X/A}$ can be viewed as the cokernel of the map $\Omega^{i-1}_{\mathbf{P}^r_A/A} \otimes \mathscr{O}_X(-m) \xrightarrow{df\wedge\cdot} \Omega^i_{\mathbf{P}^r_A/A} \otimes \mathscr{O}_X$, so the kernel of $\Omega^i_{X/A} \xleftarrow{\iota_\theta} \widetilde{\Omega}^{i+1}$ is generated by elements of the form $\iota_\theta\alpha$, $f\beta$ and $df \wedge \gamma$ (where $\alpha \in \widetilde{\Omega}^{i+2}$, $\beta \in \widetilde{\Omega}^{i+1}(-m)$, $\gamma \in \widetilde{\Omega}^i(-m)$). However by the formula $mf\beta = \iota_\theta(df \wedge \beta) + df \wedge \iota_\theta \beta$ we see that it is already generated by elements of the form $\iota_\theta \alpha$ and $df \wedge \gamma$.

As for the kernel of $\widetilde{\Omega}^{i+1} \xleftarrow{\iota_\theta + df\wedge\cdot} \widetilde{\Omega}^{i+2} \oplus \widetilde{\Omega}^i(-m)$, assume that $\iota_\theta\alpha + df \wedge \beta = 0$ ($\alpha \in \widetilde{\Omega}^{i+2}$, $\beta \in \widetilde{\Omega}^i(-m)$). Then $0 = -\iota_\theta\iota_\theta\alpha = \iota_\theta(df \wedge \beta) = mf\beta + df \wedge \iota_\theta\beta \equiv df \wedge \iota_\theta \beta \mod (f)$, from the exactness of (14) we have a $\gamma \in \widetilde{\Omega}^{i-2}(-2m)$ such that $\iota_\theta\beta \equiv df \wedge \gamma \mod (f)$, or $\iota_\theta\beta = df \wedge \gamma + f\delta$ for some $\delta \in \widetilde{\Omega}^{i-1}(-2m)$. So $0 = mf\beta + df \wedge \iota_\theta \beta = f \cdot (m\beta + df \wedge \delta)$, thus $m\beta + df \wedge \delta = 0$, or $\beta = -\frac{1}{m} df \wedge \delta$. Then it follows that $\iota_\theta \alpha = 0$, so $\alpha = \iota_\theta \eta$ for some $\eta \in \widetilde{\Omega}^{i+3}$ from (12).

Exactness elsewhere immediately follows from (12) and (13). $\square$

Notation 5. We denote the free resolution in Lemma 14 by $\Omega^i_{X/A} \leftarrow \mathscr{R}^i_\cdot$.

When $A = \mathbb{C}$, let $jac(f)$ be the ideal of $\mathbb{C}[X_0, \dots, X_r]$ generated by $\frac{\partial f}{\partial X_0}, \dots, \frac{\partial f}{\partial X_r}$, and let V_p be the $(m(p+1) - (r+1))$-degree part of $\mathbb{C}[X_0, \dots, X_r]/jac(f)$, it is shown by Griffiths[5] that there is an isomorphism $V_p \xrightarrow{\sim} H^{r-1-p.p}_{prim}(X)$ induced from the residue map. On the other hand, using the free resolution in Notation 5 we can calculate that $H^{r-1-p}_{prim}(X, \Omega^p_X)$ is naturally dual to V_p. (Note that, $H^{r-1-p}_{prim}(X, \Omega^p_X)$ is the same as $H^{r-1-p}(X, \Omega^p_X)$ unless r is odd and $p = \frac{r-1}{2}$; in this case, $H^{r-1-p}(X, \Omega^p_X)$ contains a component $H^r(\mathbf{P}^r_{\mathbb{C}}, \widetilde{\Omega}^{r+1})$, which is exactly the component generated by the hyperplane section.)

For general cases, recall that from the morphism $\kappa : X \to A$ we deduce a filtration $L^0 \supset L^1 \supset \cdots \supset L^i$ of the sheaf Ω^i_X where:

$$L^j \Omega^i_X = \mathrm{Im}(\kappa^* \Omega^j_A \otimes \Omega^{i-j}_X \to \Omega^i_X)$$

and we have $L^j / L^{j+1} = \kappa^* \Omega^j_A \otimes \Omega^{i-j}_{X/A}$. Now fix an A-free resolution $\Omega^1_A \leftarrow \mathscr{B}_\cdot$ of Ω^1_A (*i.e.* for any k, $\mathscr{B}_k$ is a free A-module), then $\mathscr{B}^j_\cdot = \bigwedge^j \mathscr{B}_\cdot$ is an A-free resolution of Ω^j_A, and $\mathscr{B}^j_\cdot \otimes_A \mathscr{R}^{i-j}_\cdot$ is a free resolution of $\kappa^*\Omega^j_A \otimes \Omega^{i-j}_{X/A}$. Consider the following complex

$$\mathscr{E}_\cdot \approx \bigoplus_{j=0}^{i} \mathscr{B}^j_\cdot \otimes_A \mathscr{R}^{i-j}_\cdot$$

where $\approx$ means that for any k we have $\mathscr{E}_k = \bigoplus_{j=0}^{i}(\mathscr{B}_\cdot^j \otimes_A \mathscr{R}_\cdot^{i-j})_k$, and the differential maps are also the same except that we should no longer regard the df in the differential maps of $\mathscr{R}_\cdot^{i-j}$ as a relative differential, but should also consider the partial derivatives of f on the coordinates of the base A. Thus df will no longer be an element in $\Gamma(\mathbf{P}_A^r, \widetilde{\Omega}(m))$, but an element in $\Gamma(\mathbf{P}_A^r, \widetilde{\Omega}(m)) \oplus \Gamma(A, \Omega_A^1) \otimes \Gamma(\mathbf{P}_A^r, \mathscr{O}_{\mathbf{P}_A^r}(m))$, and we lift it to a fixed element in $\Gamma(\mathbf{P}_A^r, \widetilde{\Omega}(m)) \oplus \Gamma(A, \mathscr{B}_0) \otimes \Gamma(\mathbf{P}_A^r, \mathscr{O}_{\mathbf{P}_A^r}(m))$.

Then the differential maps of $\mathscr{E}_\cdot$ will take $(\mathscr{B}_\cdot^j \otimes_A \mathscr{R}_\cdot^{i-j})_k$ to $(\mathscr{B}_\cdot^j \otimes_A \mathscr{R}_\cdot^{i-j})_{k-1} \oplus (\mathscr{B}_\cdot^{j+1} \otimes_A \mathscr{R}_\cdot^{i-j-1})_{k-1}$, so $\mathscr{E}_\cdot$ has a filtration $\widetilde{L}^0 \supset \widetilde{L}^1 \supset \cdots \supset \widetilde{L}^i$ where:

$$\widetilde{L}^j \mathscr{E}_\cdot \approx \bigoplus_{k=j}^{i} \mathscr{B}_\cdot^k \otimes_A \mathscr{R}_\cdot^{i-k}.$$

This filtration is compatible to the filtration $L^\cdot$ of the sheaf Ω_X^i, and $\widetilde{L}^j/\widetilde{L}^{j+1} = \mathscr{B}_\cdot^j \otimes_A \mathscr{R}_\cdot^{i-j}$ is a free resolution of L^j/L^{j+1}, so we conclude that $\mathscr{E}_\cdot$ is a free resolution of Ω_X^i.

7 Calculation on elliptic surfaces

Notation 6. Use Notation 2, and in addition

- Put $V = \mathbf{Spec}\,\mathbb{C}[t]$ and $V_\infty = \mathbf{Spec}\,\mathbb{C}[t^{-1}]$.
- Put $\mathbf{P}_V^2 = \mathbf{Proj}\,V[X,Y,Z]$ and $\mathbf{P}_{V_\infty}^2 = \mathbf{Proj}\,V_\infty[X_\infty, Y_\infty, Z_\infty]$.
- Let $E_0 \subset \mathbf{P}_V^2$ be the (maybe singular) hypersurface defined by the homogeneous polynomial $Y^2Z - 4X^3 + 3aXZ^2 - bZ^3$.
- Let $E_\infty \subset \mathbf{P}_{V_\infty}^2$ be defined by $Y_\infty^2 Z_\infty - 4X_\infty^3 + 3a_\infty X_\infty Z_\infty^2 - b_\infty Z_\infty^3$, where $a_\infty, b_\infty \in \mathbb{C}[t^{-1}]$ and $a_\infty = t^{-4n}a, b_\infty = t^{-6n}b$.

E_0 and E_∞ glues via the relation $X_\infty = t^{-2n}X$, $Y_\infty = t^{-3n}Y$ and $Z_\infty = Z$. The minimal proper regular model $\tilde{E}$ is a desingularization of $E_0 \cup E_\infty$, we denote this desingularization by $\varepsilon : \tilde{E} = \tilde{E}_0 \cup \tilde{E}_\infty \to E_0 \cup E_\infty$.

In this section we will prove Lemma 8. The element $\eta \in \Gamma(A, \mathscr{O}_A \otimes R^1\kappa_*\mathbb{C})$ is explicitly given as an element in $C^1(\mathfrak{U}, \mathscr{O}_{E_\Delta}) \oplus C^0(\mathfrak{U}, \Omega_{E_\Delta}^1)$, where $\mathfrak{U} = \{U_X, U_Y, U_Z\}$ and $U_X = \{X \neq 0\}$ *etc*. Now η is defined by

$$\eta = \eta_0^1 + \eta_1^0, \text{ where } \eta_0^1 = \begin{cases} -b\frac{Z^2}{XY} + a\frac{Z}{Y} & \text{on } U_{XY} \\ -\frac{Y}{X} & \text{on } U_{XZ} \\ 2a\frac{Z}{Y} - 4\frac{X^2}{YZ} & \text{on } U_{YZ} \end{cases}$$

and

$$\eta_1^0 = \begin{cases} \frac{1}{\Delta}(\{(-\frac{1}{6}b^3 + \frac{1}{8}ba^3)\frac{YZ^2}{X^3} + (\frac{1}{6}b^2a - \frac{3}{16}a^4)\frac{YZ}{X^2} - \frac{1}{24}ba^2\frac{Y}{X}\}d(\frac{Z}{X}) & \\ \quad +\{(\frac{1}{6}b^3 - \frac{1}{8}ba^3)\frac{Z^3}{X^3} + (-\frac{5}{12}b^2a + \frac{3}{8}a^4)\frac{Z^2}{X^2} - \frac{1}{12}ba^2\frac{Z}{X}\}d(\frac{Y}{X})) & \text{on } U_X \\ (-6b\frac{X}{Y} + 3a^2\frac{Z}{Y})\frac{XdZ - ZdX}{Y^2} & \text{on } U_Y \\ \frac{1}{\Delta}(\{-\frac{4}{3}b\frac{X^2}{Z^2} - \frac{2}{3}a^2\frac{X}{Z} + \frac{2}{3}ba\}d(\frac{Y}{Z}) + \{2b\frac{XY}{Z^2} + a^2\frac{Y}{Z}\}d(\frac{X}{Z})) & \text{on } U_Z \end{cases}.$$

For the calculation of 1. 2. of Lemma 8 (and of course the fact that η is indeed a relative cocycle), please consult [21]. Here we only represent the construction of $u \in F^1 H^2_{DR}(\tilde{E})$ in 3. of Lemma 8.

Lemma 15. *We can extend $\varepsilon^*(dt \wedge \eta)$ to a cocycle in $C^1(\mathfrak{U}, \Omega^1_{\tilde{E}_0}) \oplus C^0(\mathfrak{U}, \Omega^2_{\tilde{E}_0})$.*

Proof. The desingularization ε is achieved by a series of blow-ups. So it is obvious that $\varepsilon^* \eta^1_0$ is well-defined on $\tilde{E}_0$. As for η^0_1, it can be easily checked by hand that $dt \wedge \eta^0_1$ extends to every nonsingular point of E_0, for example on U_{YZ} we have

$$\frac{1}{\Delta}(\{-\frac{4}{3}b\frac{X^2}{Z^2} - \frac{2}{3}a^2\frac{X}{Z} + \frac{2}{3}ba\}d(\frac{Y}{Z}) + \{2b\frac{XY}{Z^2} + a^2\frac{Y}{Z}\}d(\frac{X}{Z})) = -2\frac{x}{y}dx,$$

the $\frac{1}{\Delta}$ factor is canceled. Then $\varepsilon^*(dt \wedge \eta^0_1)$ may only have poles on those fiber components disjoint to the 0-section. These fiber components, denoted by C_i, are rational curves with self-intersection $C_i^2 = -2$. By an induction using such exact sequences in the form

$$0 \to \Omega^2_{\tilde{E}_0}(\sum_{i=0}^{k} n_i C_i) \to \Omega^2_{\tilde{E}_0}((n_0+1)C_0 + \sum_{i=1}^{k} n_i C_i) \to \mathscr{O}_{C_0}(M) \to 0$$

while keeping the number $M = \sum_{i=1}^{k} n_i C_0.C_i - 2(n_0+1) \leq -1$ (this can be done because the configurations of those fiber components disjoint to the 0-section are trees), we can conclude the injectivity of the natural map $H^0(\Omega^2_{\tilde{E}_0}) \to H^0(\Omega^2_{\tilde{E}_0}(\sum n_i C_i))$ for sufficiently large n_i. Which means that any meromorphic section of $\Omega^2_{\tilde{E}_0}$ with only possible poles on C_is is always holomorphic. Hence $\varepsilon^*(dt \wedge \eta^0_1)$ can be extended to $C^0(\mathfrak{U}, \Omega^2_{\tilde{E}_0})$ and we are done. $\square$

We will need another cocycle $\alpha \in C^1(\mathfrak{U}, \Omega^1_{E_0}) \oplus C^0(\mathfrak{U}, \Omega^2_{E_0})$ defined by

$$\alpha = \alpha^1_1 + \alpha^0_2, \text{ where } \alpha^1_1 = \begin{cases} b'\frac{Z^2}{XY}dt + 3a\frac{XdZ - ZdX}{XY} & \text{on } U_{XY} \\ -2\frac{YdZ - ZdY}{XZ} & \text{on } U_{XZ} \\ -3a'\frac{Z}{Y}dt + 12\frac{X}{Y}d(\frac{X}{Z}) & \text{on } U_{YZ} \end{cases}$$

and

$$\alpha^0_2 = \begin{cases} (-\frac{1}{6}b'\frac{Z^2}{X^2} + \frac{1}{2}a'\frac{Z}{X})\frac{YdZ - ZdY}{X^2} \wedge dt + \frac{1}{2}a\frac{Z}{X}d(\frac{Y}{X}) \wedge d(\frac{Z}{X}) & \text{on } U_X \\ (9ba' - 6b'a)\frac{Z^2}{Y^2}d(\frac{Z}{Y}) \wedge dt + (12b'\frac{XZ}{Y^2} - 9aa'\frac{Z^2}{Y^2})d(\frac{X}{Y}) \wedge dt & \\ \quad +(36b\frac{X}{Y} - 18a^2\frac{Z}{Y})d(\frac{X}{Y}) \wedge d(\frac{Z}{Y}) & \text{on } U_Y \\ 0 & \text{on } U_Z \end{cases}.$$

Consider the transformation $X \mapsto p^2X$, $Y \mapsto p^3Y$, $Z \mapsto Z$, $a \mapsto p^4a$ and $b \mapsto p^6b$, for some polynomial p. Under this transformation we can calculate that $\alpha^1_1 \mapsto p\alpha^1_1 - 6p'\eta^1_0 \wedge dt$ and $\alpha^0_2 \mapsto p\alpha^0_2 - 6p'\theta^0_1 \wedge dt$, where

$$\theta^0_1 = \begin{cases} \frac{1}{6}b\frac{Z^2}{X^2}\frac{YdZ - ZdY}{X^2} - \frac{1}{4}a\frac{YZ}{X^2}d(\frac{Z}{X}) + \frac{1}{2}a\frac{Z^2}{X^2}d(\frac{Y}{X}) & \text{on } U_X \\ (3a^2\frac{Z}{Y} - 6b\frac{X}{Y})\frac{XdZ - ZdX}{Y^2} & \text{on } U_Y \\ 0 & \text{on } U_Z \end{cases}.$$

So obviously $(p\alpha_1^1 - 6p'\eta_0^1 \wedge dt) + (p\alpha_2^0 - 6p'\theta_1^0 \wedge dt)$ is also a cocycle.

Notation 7. For $p,q \in \mathbb{C}(t)$, define $\xi(p,q) = (p\alpha_1^1 - 6p'\eta_0^1 \wedge dt + q\eta_0^1 \wedge dt) + (p\alpha_2^0 - 6p'\theta_1^0 \wedge dt + q\eta_1^0 \wedge dt)$.

We can parallelly define ξ_∞ and verify that $\xi_\infty(p_\infty, q_\infty) = \xi(t^{-n}p_\infty, -t^{-n-2}q_\infty)$.

Notation 8. Put $\omega = \frac{dx}{y}$ and $\tilde{\omega} = \omega \wedge dt$.

We also need cochains $\psi_a, \psi_b \in C^0(\mathfrak{U}, \Omega^1_{\tilde{E}_0})$ where

$$\psi_a = \begin{cases} \{\frac{1}{4}ba\frac{YZ^2}{X^3} - \frac{3}{8}a^2\frac{YZ}{X^2} + b\frac{Y}{X}\}d(\frac{Z}{X}) - \{\frac{1}{4}ba\frac{Z^3}{X^3} - \frac{3}{4}a^2\frac{Z^2}{X^2} + b\frac{Z}{X}\}d(\frac{Y}{X}) & \\ \quad +\{\frac{1}{8}b'a\frac{YZ^3}{X^4} - \frac{3}{8}aa'\frac{YZ^2}{X^3} + \frac{1}{2}b'\frac{YZ}{X^2}\}dt & \text{on } U_X \\ \{-18ba'\frac{X^2Z}{Y^3} + 6bb'\frac{XZ^2}{Y^3} + 2b'\frac{X}{Y}\}dt - \{36ba\frac{X^2}{Y^2} - 18b^2\frac{XZ}{Y^2}\}d(\frac{Z}{Y}) & \\ \quad +\{36ba\frac{XZ}{Y^2} - 18b^2\frac{Z^2}{Y^2} + 12b\}d(\frac{X}{Y}) & \text{on } U_Y \\ 4\frac{X}{Z}d(\frac{Y}{Z}) - 6\frac{Y}{Z}d(\frac{X}{Z}) & \text{on } U_Z \end{cases}$$

and

$$\psi_b = \begin{cases} \{-\frac{1}{4}b'a\frac{YZ^2}{X^3} + \frac{3}{4}aa'\frac{YZ}{X^2}\}dt - \{\frac{1}{2}ba\frac{YZ}{X^2} - \frac{3}{4}a^2\frac{Y}{X}\}d(\frac{Z}{X}) & \\ \quad +\{\frac{1}{2}ba\frac{Z^2}{X^2} - \frac{3}{2}a^2\frac{Z}{X}\}d(\frac{Y}{X}) & \text{on } U_X \\ \{36a^3\frac{XZ}{Y^2} - 18ba^2\frac{Z^2}{Y^2} + 12a^2\}d(\frac{X}{Y}) - \{36a^3\frac{X^2}{Y^2} - 18ba^2\frac{XZ}{Y^2}\}d(\frac{Z}{Y}) & \\ \quad -\{18a^2a'\frac{X^2Z}{Y^3} - 6b'a^2\frac{XZ^2}{Y^3} - 3aa'\frac{X}{Y}\}dt & \text{on } U_Y \\ -3a'\frac{Y}{Z}dt - (8\frac{X^2}{Z^2} - 4a)d(\frac{Y}{Z}) + 12\frac{XY}{Z^2}d(\frac{X}{Z}) & \text{on } U_Z \end{cases}.$$

Regard the Čech complex of $\Omega^i_{\tilde{E}}$ on the minimal proper regular model $\tilde{E}$ as the bicomplex $\mathscr{C}^\cdot(\tilde{E}, \Omega^i_{\tilde{E}})$:

$$C^\cdot(\mathfrak{U}, \Omega^i_{\tilde{E}_0}) \oplus C^\cdot(\mathfrak{U}_\infty, \Omega^i_{\tilde{E}_\infty}) \to C^\cdot(\mathfrak{U} \cap \mathfrak{U}_\infty, \Omega^i_{\tilde{E}_0 \cap \tilde{E}_\infty})$$

where $\mathfrak{U}_\infty = \{U_{X_\infty}, U_{Y_\infty}, U_{Z_\infty}\}$, $\mathfrak{U} \cap \mathfrak{U}_\infty = \{U_{XX_\infty}, U_{YY_\infty}, U_{ZZ_\infty}\}$, $U_{XX_\infty} = U_X \cap U_{X_\infty}$ and $U_{X_\infty} = \{X_\infty \neq 0\}$ *etc.*

Definition 4. Fix $p,q \in \mathbb{C}[t]$ and $p_\infty, q_\infty \in \mathbb{C}[t^{-1}]$. Put $g = p - t^{-n}p_\infty$ and $h = q - t^{-n-2}q_\infty$. Assume that g and $h - 6g'$ are divisible by Λ, and take $\sigma, \tau \in \mathbb{C}[t, t^{-1}]$ to be such that

$$\begin{pmatrix} a & b \\ \frac{3}{2}a' & b' \end{pmatrix} \begin{pmatrix} \sigma \\ \tau \end{pmatrix} = \begin{pmatrix} g \\ h - 6g' \end{pmatrix}.$$

Then we define $u \in F^1 H^2_{DR}(\tilde{E})$ as

$$u = (\xi - \lfloor r \rfloor \tilde{\omega}) \oplus (\xi_\infty + (r - \lfloor r \rfloor)\tilde{\omega}) \oplus (\sigma\psi_a + \tau\psi_b)$$

where $\xi = \xi(p,q)$, $\xi_\infty = \xi_\infty(p_\infty, -q_\infty) = \xi(t^{-n}p_\infty, t^{-n-2}q_\infty)$ and $r = -\frac{12}{\Lambda^2}\rho(g,h)$.

Lemma 16. *The cocycle u in Definition 4 is well-defined.*

Proof. Since $\det\begin{pmatrix} 2a & b \\ 3a' & b' \end{pmatrix} = \frac{1}{2}\Lambda$ and we have assumed that $g, h-6g'$ are divisible by Λ, the $\sigma, \tau \in \mathbb{C}[t,t^{-1}]$ can be actually taken. Note the identity

$$-6b\sigma' - 6a^2\tau' - 5b'\sigma - \frac{21}{2}aa'\tau = -\frac{12}{\Lambda^2}\rho(g,h),$$

so we have $r \in \mathbb{C}[t,t^{-1}]$. Now $\varepsilon^*(\xi - \lfloor r \rfloor \tilde{\omega})$ extends to an element in $C^1(\mathfrak{U}, \Omega^1_{\tilde{E}_0}) \oplus C^0(\mathfrak{U}, \Omega^2_{\tilde{E}_0})$, meanwhile $\varepsilon^*(\xi_\infty + (r - \lfloor r \rfloor)\tilde{\omega})$ extends to an element in $C^1(\mathfrak{U}_\infty, \Omega^1_{\tilde{E}_\infty}) \oplus C^0(\mathfrak{U}_\infty, \Omega^2_{\tilde{E}_\infty})$, and $\varepsilon^*(\sigma\psi_a + \tau\psi_b) \in C^0(\mathfrak{U} \cap \mathfrak{U}_\infty, \Omega^1_{\tilde{E}_0 \cap \tilde{E}_\infty})$. For u to be a cocycle, we should check that $\xi - \xi_\infty - r\tilde{\omega} = (\delta + d)(\sigma\psi_a + \tau\psi_b)$. This is obtained from the following facts:

$$\delta\psi_a = a\alpha_1^1 + \frac{3}{2}a'\eta_0^1 \wedge dt \tag{15}$$
$$\delta\psi_b = b\alpha_1^1 + b'\eta_0^1 \wedge dt \tag{16}$$
$$d\psi_a = a\alpha_2^0 + \frac{15}{2}a'\eta_1^0 \wedge dt - 6a'\theta_1^0 \wedge dt + 5b'\tilde{\omega} \tag{17}$$
$$d\psi_b = b\alpha_2^0 + 7b'\eta_1^0 \wedge dt - 6b'\theta_1^0 \wedge dt + \frac{21}{2}aa'\tilde{\omega} \tag{18}$$
$$dt \wedge \psi_a = 6a\eta_1^0 \wedge dt - 6a\theta_1^0 \wedge dt + 6b\tilde{\omega} \tag{19}$$
$$dt \wedge \psi_b = 6b\eta_1^0 \wedge dt - 6b\theta_1^0 \wedge dt + 6a^2\tilde{\omega} \tag{20}$$

And these formulae above are checked by a computer (cf. [21]). □

Lemma 17. *Let u be as in Definition 4. Then $u|_{E_\Delta} \simeq \upsilon \wedge dt$ where*

$$\upsilon = (-\frac{1}{2}\frac{\Delta'}{\Delta}p - 6p' + q)\eta + (\frac{1}{2}\frac{a\Lambda}{\Delta}p - \lfloor r \rfloor)\omega.$$

Proof. At first we have $u|_{E_\Delta} \simeq \xi - \lfloor r \rfloor \tilde{\omega}$. Then put $\psi = \frac{1}{\Delta}(a^2\psi_a - b\psi_b)$ and calculate

$$\xi - (\delta + d)(p\psi) = (-\frac{1}{2}\frac{\Delta'}{\Delta}p - 6p' + q)\eta \wedge dt + \frac{1}{2}\frac{a\Lambda}{\Delta}p\tilde{\omega}$$

so we are done. □

Finally note that in the above arguments we can always replace p_∞ and q_∞ by $p_\infty + \zeta_\infty$ and $q_\infty + \lambda_\infty$, with $\deg\zeta_\infty$ and $\deg\lambda_\infty$ sufficiently small, to make g and $h - 6g'$ divisible by Λ. This will not affect the expression of υ. So the assumption that g and $h - 6g'$ are divisible by Λ in Definition 4 is not essential, we have proved 3. of Lemma 8.

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