THE MATHEMATICAL SOLUTION OF A CELLULAR AUTOMATON MODEL WHICH SIMULATES TRAFFIC FLOW WITH A SLOW-TO-START EFFECT

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Abstract

In this paper we investigate a cellular automaton model associated with traffic flow and of which the mathematical solution is unknown before. We classify all kinds of stationary states and show that every state finally evolves to a stationary state. The obtained flow-density relation shows multiple branches corresponding to the stationary states in congested phases, which are essentially due to the slow-to-start effect introduced into this model. The stability of these states is formulated by a series of lemmas, and an algorithm is given to calculate the stationary state that the current state finally evolves to. This algorithm has a computational requirement in proportion to the number of cars.

Key words: cellular automaton, traffic flow, flow-density, multiple branches, metastable states
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1. Introduction

Cellular automata[1] (CA) provide a simple, flexible way for modelling and are suitable for computer simulations. Many interesting phenomena can be observed in such simulation and then leave challenges to mathematicians. In order to study traffic flow, CA have been used extensively in recent years, and many traffic CA models have been proposed so far.[2-7]

There are basically two types of traffic CA models: Euler form and Lagrange form.[12] Models in Euler form, such as the Burger's CA[8,10] which can be derived from Burger's equation using an ultradiscrete method[9], focus on the number of cars at each site; while the Lagrange form models or the car-following models, such as the Nagel-Schreckenberg (NS) model[3], focus rather on the headway and velocity of each car. These two types of representations are joined

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with an Euler-Lagrange transformation,[11,12] which is a discrete version of the well-known one in hydrodynamics.

Usually the flow-density relation, or the so-called "fundamental diagram" of each traffic CA is calculated by computer simulations, and is compared with the measurements of real traffic. We show an example of real data[15] at the left of Fig.1, and point out that there is a wide scattering area near the critical point where the transition from free phase to congested phase occurs. This area suggests that there are multiple metastable states around the critical density. The flow-density relation of the model we investigate is drawn at right, which shows multiple branches really form the skeleton of such area.

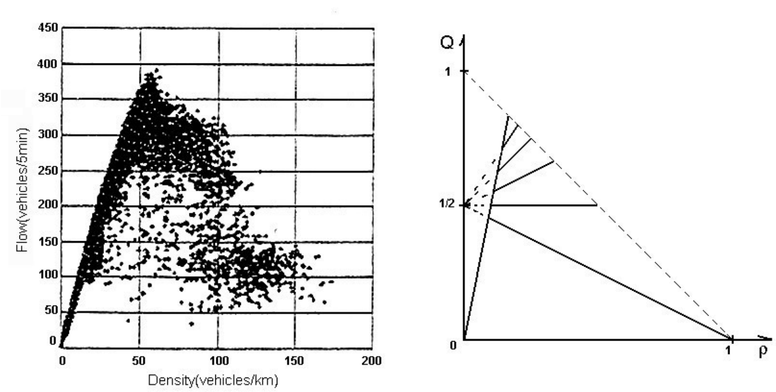

Figure 1: Left: An observed flow-density relation at the Tomei expressway in Japan. Right: The flow-density relation of the model for $M = 5$.

The model we investigate in this paper is in Lagrange form and written as:

$$x_i^{t+1} = x_i^t + \min(M, x_{i+1}^t - x_i^t - 1, x_{i+1}^{t-1} - x_i^{t-1} - 1) \qquad (1)$$

where M is a constant and x_i^t denotes the position of the ith car at time t. As this is a CA model, the space and time are both discrete so $t \in \mathbf{N}$, $x_i^t \in \mathbf{Z}$. And we consider a periodic boundary condition or traffic flow on a circuit, which means that the position x is identical with the position $x + L$, and that the ith car is identical with the $(i + N)$th car, where L and N represent the length of the circuit and the number of cars respectively. The constant M can be understood as a legal limitation of the velocity, the term $x_{i+1}^t - x_i^t - 1$ avoids a collision, and the term $x_{i+1}^{t-1} - x_i^{t-1} - 1$ represents the inertia of the car or the reaction delay of the driver, which means that if $x_{i+1}^{t-1} - x_i^{t-1} - 1 < x_{i+1}^t - x_i^t - 1 \leq M$, then the car will maintain a low speed for an extra time step (slow-to-start effect). This kind of rule first appears in a slow-start (SlS) model proposed by Takayasu and Takayasu,[2] which is the first known deterministic two-value CA to show metastable states, its generalizations given by Nishinari both in Euler[13] and Lagrange[12] form show metastable states and multiple branches in the fundamental diagram. These multiple branches are really characteristic, which appear in the model combined with the slow-to-start effect,[14] but are rarely observed in other models.[3,4,5] We will prove in section 2 that this effect results in many kinds of congested phases, and these phases form the multiple branches in the fundamental diagram.

Very often, a traffic CA also takes in a driver's perspective or anticipation, which means that the ith car's behavior depends not only on the $(i+1)$th car, but also the $(i+2)$th car and even the $(i+3)$th car. The slow-to-start effect and the driver's anticipation are somehow cancelling each other, for although the driver cannot response to the fluctuation of his headway immediately, he can possibly predict the fluctuation through the headway of the car before him. So, as model (1) only takes in the slow-to-start effect, it is not amazing to find an extreme behavior of cars, which makes a jam very easy to remain and very hard to dissolve. This will be made clear in section 3 where we show some results about a jam's "stability". The character of such results have been observed in a model combined with the slow-to-start effect,[14] through computer simulations. There is also an algorithm to predict the final stationary state which has a computational requirement in proportion to N^2 (where N is the number of cars) given in section 3.

In section 4 we will make good use of the facts proved in section 3, and develop some tools to investigate the detailed behavior of a jam's remaining. As a result, an algorithm with a computational requirement in proportion to N is obtained.

We conclude this paper in section 5 with a generalization of model (1) to include a driver's perspective. Some elemental discussions about the generalized model are given.

2. Stationary states and the flow-density relation

Notation2.1. By C_i we mean the ith car and C_i^t the ith car at time t. $x_i^{t+1} - x_i^t$ is called the **velocity** of C_i^t and denoted by v_i^t. $x_{i+1}^t - x_i^t - 1$ is called the **headway** of C_i^t and denoted by h_i^t. Eq.(1) is then rewritten as:

$$v_i^t = \min(M, h_i^t, h_i^{t-1}) \tag{2}$$

with $h_i^{t+1} = h_i^t + v_{i+1}^t - v_i^t$ by definition. Immediately we get $h_i^t \geq v_i^t, h_i^{t-1} \geq v_i^t$.

There are two apparent types of stationary states which satisfy this equation, namely the "free state" that $v_i^t = M, h_i^t \geq M$ for all i; and the "τ-uniform state" that $v_i^t = h_i^t = \tau < M$ for all i. Note that if we have a uniform state at time t, which means $v_i^t = h_i^t$ for all i, then $h_i^{t-1} \geq v_i^t = h_i^t$ for all i, however $\sum_{i=1}^N h_i^t = L - N$ is a constant, so we conclude that all the inequalities should be equalities and the state at time $t-1$ must also be a uniform state. This shows that the uniform state is very unstable.

Lemma 2.2. *The following inequalities hold:*

(i) $h_i^{t+1} \geq v_{i+1}^t$

(ii) $h_i^{t+1} \geq \min(M, h_{i+1}^t, h_{i+1}^{t-1})$
(iii) $v_i^{t+1} \geq \min(v_{i+1}^t, v_i^t)$

Proof. (i)

$$\begin{aligned} h_i^{t+1} &= h_i^t + v_{i+1}^t - v_i^t \text{ (by definition)} \\ &\geq v_{i+1}^t \text{ (for } h_i^t \geq v_i^t) \end{aligned}$$

(ii) by (i) we have $h_i^{t+1} \geq v_{i+1}^t = \min(M, h_{i+1}^t, h_{i+1}^{t-1})$.
(iii)

$$\begin{aligned} v_i^{t+1} &= \min(M, h_i^{t+1}, h_i^t) \\ &\geq \min(M, v_{i+1}^t, h_i^t) \text{ (for } h_i^{t+1} \geq v_{i+1}^t \text{ by (i))} \\ &\geq \min(v_{i+1}^t, \min(M, h_i^t, h_i^{t-1})) \\ &= \min(v_{i+1}^t, v_i^t) \text{ (by Eq.(2))} \end{aligned}$$

Corollary 2.3. $\min\{v_i^t | i \in \boldsymbol{Z}\}$ *(the minimal velocity of cars at time t) is a nondecreasing function of t.*

Proof. Use (iii) of lemma 2.2.

Briefly speaking, this corollary shows that there is no spontaneous jam formation in model(1).

Notation2.4. We use τ_t to denote this $\min\{v_i^t | i \in \mathbf{Z}\}$.

Obviously, the slow-to-start effect appears in an accelerating process and result in a nontrivial low speed. The definition below focuses on this phenomenon and clarifies the main object we investigate in this paper.

Definition2.5. If one of the following conditions is satisfied, we call C_i^t an α-**front** for $0 \leq \alpha < M$.

(i) $v_i^t = h_i^t = \alpha, v_{i+1}^t > \alpha$
(ii) $v_i^t = \alpha, h_i^t > \alpha$

Note that the two conditions are incompatible. We say the front is **type A** when (i) is satisfied and **type B** when (ii) is satisfied.

Lemma 2.6. *If C_i^t is an α-front, then C_{i+1}^{t-1} or C_i^{t-1} is a β-front with $\beta \leq \alpha$. More precisely, when C_i^t is a type A α-front, C_{i+1}^{t-1} must be a type B γ-front with $\gamma \leq \alpha$. In the case C_i^t is type B, we have C_i^{t-1} as a type A α-front or a type B δ-front with $\delta < \alpha$.*

Proof. (i) When C_i^t is type A:
Let $\gamma = v_{i+1}^{t-1}$, and we have $\alpha = h_i^t \geq v_{i+1}^{t-1} = \gamma$, $h_{i+1}^{t-1} \geq v_{i+1}^t > \alpha \geq \gamma$, so C_{i+1}^{t-1} is a type B γ-front.

(ii) When C_i^t is type B:
We have $v_i^t = \alpha < M, h_i^t > v_i^t$ so $v_i^t = \alpha$ must be equal to h_i^{t-1} since $v_i^t = \min(M, h_i^t, h_i^{t-1})$. Then (a) $v_i^{t-1} = \alpha$ or (b) $v_i^{t-1} < \alpha$. If $v_i^{t-1} = \alpha$, then $\alpha < h_i^t = h_i^{t-1} - v_i^{t-1} + v_{i+1}^{t-1} = v_{i+1}^{t-1}$ and we can say C_i^{t-1} is a type A α-front. If $\delta = v_i^{t-1} < \alpha$, the condition for C_i^{t-1} to be a type B δ-front is then satisfied.

Definition2.7. Let F_t be the set consists of all fronts at time t. Lemma 2.6 suggests that we can define a **front map** $\phi_t : F_t \to F_{t-1}$ like this:
For $C_i^t \in F_t$, let $\phi_t(C_i^t) = \begin{cases} C_{i+1}^{t-1} & \text{if } C_i^t \text{ is type A} \\ C_i^{t-1} & \text{if } C_i^t \text{ is type B} \end{cases}$
We also define G_t to be the set consists of all τ_t-fronts at time t.

An example is shown in Fig.2 with $L = 40$,$N = 9$,$M = 4$, from time point 0 to 10. We can see the concept “front” somehow corresponds to “the front of a jam”, and the front map somehow joins the fronts of “the same jam”.

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00: __1______2____A_A__5_____A__7______A9___
01: 9_____1______2_B__B____5___B____7__B____
02: ____9_____1___A_B___4_____A__6____A8____
03: ________9____1_B__B_____4___B____6B____8
04: ___8________9_A_B____3_____4__5__A7_____
05: _______8_____9_B__2______3___4__5B____7_
06: __7________8__A_B_____2_____3__4A6______
07: ______7______8_B__1_______2___34B____6__
08: _6________7___A_B_____1______23A5_______
09: _____6_______7_B__9_______1__23B____5___
10: 5________6____A_B_____9_____12A4________
```

Figure 2: An example. The 9 cars are denoted by numbers from 1 to 9, or by alphabet A (resp. B) when it is a type A (resp. type B) front.

Lemma 2.8. $\phi_t|_{G_t} : G_t \to F_{t-1}$ *(the restriction of ϕ_t to G_t) is an injection.*

Proof. If this is not true, we can assume that there exist C_i^t and C_{i+1}^t in G_t such that $\phi_t(C_i^t) = \phi_t(C_{i+1}^t)$. That means C_i^t is a type A τ_t-front and C_{i+1}^t a type B τ_t-front. C_i^t to be a type A τ_t-front implies $v_{i+1}^t > \tau_t$ while C_{i+1}^t to be a type B τ_t-front implies $v_{i+1}^t = \tau_t$. That is a contradiction.

Note that if $\tau_t = \tau_{t-1}$, then by lemma 2.6 the image of $\phi_t|_{G_t}$ lies in G_{t-1}. τ_t is a nondecreasing function of t by Corollary 2.3, and it is an integer less than M, so it becomes a constant when $t \gg 0$. We write the constant τ_∞. If $\tau_\infty = M$, the traffic is free. If $\tau_\infty < M$, G_t with $t \gg 0$ is empty precisely when $h_i^t = v_i^t = \tau_\infty$ for all i. This uniform state is trivial so we assume that $G_t \neq \emptyset$. Then lemma 2.8 implies that $\#G_t$ (the cardinality of G_t) is non-increasing when $t \gg 0$. That means we can assume $\#G_t$ to be a positive constant when $t \gg 0$.

Definition2.9. Let C_i^t be a τ-front. The **preceding τ-front** of C_i^t is the τ-front C_j^t with $j > i$ such that no other τ-front exists between C_i^t and C_j^t. We say C_i^t **connects properly** to a τ-front C_j^t if for all $i < k < j$, the following conditions are satisfied:

1. If C_i^t is type B, then $h_i^t \leq M$ and C_{i+1}^t is *not* a type B τ-front.
2. If $v_{k+1}^t = \tau$, then the following (a) *or* (b) holds:
   (a) $\tau < h_k^{t+1} = h_k^t + \tau - v_k^t \leq M$ and $h_{k+1}^t = \tau$.
   (b) $h_k^t = v_k^t \geq \tau$.
3. If $v_{k+1}^t \neq \tau$, then $h_k^t + \tau - v_k^t = M$.
4. If $k > i+1$ *or* C_i^t is type B, then v_k^t should be M unless the (2b) case.

Lemma 2.10. *If a τ-front C_i^t connects properly to a τ-front C_j^t, then $v_k^t \geq \tau$ for all $i < k < j$, and C_j^t is the preceding front of C_i^t.*

Proof. When $k = i+1$ and C_i^t is type A, then $v_k^t > \tau$ by definition. In the case $k > i+1$ or C_i^t is type B, condition (4) and (2b) implies $v_k^t \geq \tau$. Now we assume $v_k^t = \tau$. Condition (4) says that this could happen only if $v_k^t = h_k^t = \tau = v_{k+1}^t$. So C_k^t is not a τ-front.

Roughly speaking, a proper connection structure looks like several successive cars with headway τ and velocity τ preceding several successive cars with headway $2M-\tau$ and velocity M. The next lemma shows that this structure somehow repeats itself after two time steps.

Lemma 2.11. *If a τ-front C_i^t connects properly to its preceding τ-front C_j^t, then for all $i \leq k < j$, we have $x_k^{t+2} = x_{k+1}^t + \tau - 1, v_k^{t+2} = v_{k+1}^t$. Moreover, if C_p^{t+1} and C_q^{t+1} are τ-fronts satisfy $\phi_{t+1}(C_p^{t+1}) = C_i^t, \phi_{t+1}(C_q^{t+1}) = C_j^t$, then C_p^{t+1} connects properly to C_q^{t+1}.*

Proof. First we use (iii) of lemma 2.2 to get $v_k^{t+1} \geq \min(v_{k+1}^t, v_k^t) \geq \tau$. We prove the first part of this lemma by dividing it into the following cases. In each case we basically calculate h_k^{t+1} by $h_k^{t+1} = h_k^t - v_k^t + v_{k+1}^t$ and v_k^{t+1} by Eq.(2). $x_k^{t+2} = x_{k+1}^t + \tau - 1$ holds precisely when $v_k^t + v_k^{t+1} = h_k^t + \tau$. Then we estimate h_k^{t+2} by $h_k^{t+2} = h_k^{t+1} - v_k^{t+1} + v_{k+1}^{t+1} \geq h_k^{t+1} - v_k^{t+1} + \tau$, and get v_k^{t+2} by Eq.(2).

(i) $k = i$ and C_i^t is type A:
Note the definition of a type A front and we have $h_i^{t+1} = v_{i+1}^t, v_i^{t+1} = \tau$ and $v_i^t + v_i^{t+1} = h_i^t + \tau$ can be checked. $h_i^{t+2} \geq h_i^{t+1}$ so $v_i^{t+2} = h_i^{t+1} = v_{i+1}^t$.

(ii) $k = i$ and C_i^t is type B:
The definition of a type B front implies $h_i^{t+1} = h_i^t + v_{i+1}^t - \tau \geq h_i^t$, condition (1) says $M \geq h_i^t$, so $v_i^{t+1} = h_i^t$ and $v_i^t + v_i^{t+1} = h_i^t + \tau$ can be checked. If $i+1 = j$, then C_j^t must be type A, so we have $v_{i+1}^t = v_{i+1}^{t+1} = \tau$, then $h_i^{t+1} = h_i^t, h_i^{t+2} = v_i^{t+2} = \tau = v_{i+1}^t$. If $i+1 < j$, by condition (4), $v_{i+1}^t = M$ or C_{i+1}^t satisfies (2b). In the former case, $h_i^{t+1} \geq M, h_i^{t+2} \geq M$ so $v_i^{t+2} = M = v_{i+1}^t$. In the latter case, we have $v_{i+1}^{t+1} = \tau$ so $h_i^{t+2} = v_{i+1}^t$ and $v_i^{t+2} = h_i^{t+2} = v_{i+1}^t$.

(iii) $k \geq i+1, v_{k+1}^t = \tau$:
If condition (2a) is satisfied, we have $h_k^{t+1} \leq M$ and $h_k^{t+1} \leq h_k^t$ since $v_k^t \geq \tau$. So $v_k^{t+1} = h_k^{t+1} = h_k^t + \tau - v_k^t$. From $h_{k+1}^t = \tau$ and $v_{k+2}^t \geq \tau$ we calculate $v_{k+1}^{t+1} = \tau$ so $h_k^{t+2} = \tau$ and $v_k^{t+2} = \tau = v_{k+1}^t$. If condition (2b) is satisfied, we have $h_k^{t+1} = \tau$ and $v_k^{t+1} = \tau$; $h_k^{t+2} \geq \tau$ so $v_k^{t+2} = \tau = v_{k+1}^t$.

(iv) $k \geq i+1, v_{k+1}^t \neq \tau$:
Condition (4) says that $v_{k+1}^t = M$ or C_{k+1}^t satisfies (2b). In the former case we have $h_k^{t+1} = 2M - \tau$ by condition (3), then $v_k^{t+1} = M$. $h_k^{t+2} \geq M$ so $v_k^{t+2} = M = v_{k+1}^t$. In the latter case, $h_k^{t+1} = M - \tau + v_{k+1}^t \geq M$ and $v_k^{t+1} = M$. Since C_{k+1}^t satisfies (2b), we have just showed in (iii) that $v_{k+1}^{t+1} = \tau$, which implies $h_k^{t+2} = v_{k+1}^t$ and so $v_k^{t+2} = v_{k+1}^t$.

To prove the second part of this lemma, we simply check the four conditions for all $p < k < q$. (1) If C_p^{t+1} is type B, that means $p = i$ and C_i^t is type A by lemma 2.6. So $h_p^{t+1} = v_{i+1}^t \leq M$. If $v_{i+1}^{t+1} = \tau$, then h_{i+1}^{t+1} must be τ since $h_{i+1}^t \geq v_{i+1}^t > \tau$ and $\tau = v_{i+1}^{t+1} = \min(M, h_{i+1}^t, h_{i+1}^{t+1})$. That means C_{p+1}^{t+1} cannot be a type B τ-front. (2)(3)(4) As $p < k < q$ implies $i \leq k < j$, we summarize the preceding calculation results here: If (ii) and $i+1 = j$, we have $v_k^{t+1} = h_k^{t+1}$ and $v_{k+1}^{t+1} = \tau$; If (ii) and $i+1 < j$, we have $h_k^{t+1} - v_k^{t+1} + \tau = v_{k+1}^t$, with $v_{k+1}^t = M$ or $v_{k+1}^{t+1} = \tau = h_{k+1}^{t+1}$. If (iii), we have $v_k^{t+1} = h_k^{t+1}$ and $v_{k+1}^{t+1} = \tau$. If (iv), we have $v_k^{t+1} = M, h_k^{t+1} - v_k^{t+1} + \tau = v_{k+1}^t$, with $v_{k+1}^t = M$ or $v_{k+1}^{t+1} = \tau = h_{k+1}^{t+1}$. (Note that C_i^t must be type B when $k = i$, and we can assume $h_{k+1}^t = \tau$ if $v_{k+1}^t = \tau$, for otherwise C_{k+1}^t is a type B τ-front so we have $k+1 = j$ and $k = q = j-1$). In any case, (2)(3)(4) hold.

Definition2.12. A **τ-congested state** is a state that contains at least one τ-front and all its τ-fronts connect properly to their preceding τ-fronts.

Corollary 2.13. *If we have a τ-congested state at time t, then the state at time $t+1$ is also a τ-congested state, and $x_i^{t+2} = x_{i+1}^t + \tau - 1, v_i^{t+2} = v_{i+1}^t$ for all i.*

Proof. Take an arbitrary τ-front C_i^t. By lemma 2.11, what to prove is that we can find a τ-front C_p^{t+1} satisfies $\phi_{t+1}(C_p^{t+1}) = C_i^t$. This can always be done when C_i^t is type A since then C_i^{t+1} is automatically a type B τ-front. Now assume C_i^t to be type B. C_i^t is properly connected, so we can apply condition (2) to C_{i-1}^t and get $h_{i-1}^t = v_{i-1}^t \geq \tau$, that means $h_{i-1}^{t+1} = v_{i-1}^{t+1} = \tau$. On the other hand, lemma 2.10 says $v_{i+1}^t \geq \tau$, so $h_i^{t+1} = h_i^t - \tau + v_{i+1}^t \geq h_i^t > \tau$, and hence $v_i^{t+1} > \tau$. Now we can say that C_{i-1}^{t+1} is a type A τ-front.

Using the density $\rho = N/L$, we can represent the flow Q of a τ-congested state by ρ as:
$Q = \frac{1}{2L}\sum_{i=1}^{N}(x_i^{t+2} - x_i^t) = \frac{1}{2L}\sum_{i=1}^{N}(x_{i+1}^t - x_i^t + \tau - 1) = \frac{N(\tau-1)+L}{2L} = \frac{\tau-1}{2}\rho + \frac{1}{2}$.
Since the headway of a car in a τ-congested state is always between τ and $2M - \tau$, we have $\frac{1}{2M-\tau+1} < \rho < \frac{1}{\tau+1}$.

Theorem 2.14. *Every state finally evolves to one of the followings:*

(i) *a free state.*
(ii) *a τ-uniform state.*
(iii) *a τ-congested state.*

Proof. As we have discussed above, if not the (i) or (ii) case, we can assume that $\tau_t = \tau_\infty = \tau$ and $\#G_t$ is a positive constant at time $t \gg 0$. Now we prove it to be the (iii) case. Since $\#G_t$ is a constant, if we have a type B τ-front at time t there must be a corresponding type A τ-front at time $t+1$ (and vice versa). Then by lemma 2.11, once at time t a τ-front C_i^t connects properly to its preceding τ-front, there will always be a τ-front corresponds to C_i^t which also connects properly to its preceding τ-front after t. This verifies that we can only consider a type A τ-front C_i^t and its correspondences at time $t+2, t+4$, *etc.* to see if it connects properly to its preceding τ-front at last. We enumerate some cases at the beginning in which the type A τ-front C_i^t connects to its preceding just properly:

- $h_{i+1}^t \le M + v_{i+1}^t - \tau, v_{i+2}^t = \tau$.
  In this case, if $h_{i+1}^t = v_{i+1}^t$, then the condition (2b) in definition 2.9 is satisfied for $k = i+1$; otherwise if $h_{i+1}^t > v_{i+1}^t$, we have $h_{i+2}^t = \tau$ since the type B τ-front C_{i+2}^t will not have a correspondence at time $t+1$ if $h_{i+2}^t > \tau$, so the condition (2a) in definition 2.9 is satisfied for $k = i+1$.

- $h_{i+1}^t = M + v_{i+1}^t - \tau, v_{i+2}^t = h_{i+2}^t, v_{i+3}^t = \tau$.
  In this case condition (3) in definition 2.9 is held for $k = i+1$.

- $h_{i+1}^t = M + v_{i+1}^t - \tau, v_{i+2}^t = M, h_{i+2}^t \le 2M - \tau, v_{i+3}^t = \tau$.
  In this case if $h_{i+2}^t > v_{i+2}^t$, then h_{i+3}^t must be τ, for otherwise the type B τ-front C_{i+3}^t will not have a correspondence at time $t+1$. So condition (2a) is satisfied for $k = i+2$.

- $h_{i+1}^t = M + v_{i+1}^t - \tau$; $j \ge i+2, v_k^t = M, h_k^t = 2M - \tau$ for all $i+2 \le k \le j$; $v_{j+1}^t = h_{j+1}^t, v_{j+2}^t = \tau$.

- $h_{i+1}^t = M + v_{i+1}^t - \tau$; $j \ge i+2, v_k^t = M, h_k^t = 2M - \tau$ for all $i+2 \le k \le j$; $v_{j+1}^t = M, h_{j+1}^t \le 2M - \tau, v_{j+2}^t = \tau$.

The last two are similar to the above. Now before considering C_i^t in the following cases, let f denote v_{i+1}^t, and confirm that C_i^t is correspondent to $C_{i-1}^{t+2}, C_{i-2}^{t+4}, C_{i-3}^{t+6}, \ldots$ with $v_{i-1}^{t+2} = v_{i-2}^{t+4} = v_{i-3}^{t+6} = \ldots = f$.

(i) $h^t_{i+1} > M + f - \tau$:
We have $h^{t+1}_{i+1} \geq h^t_{i+1} + \tau - v^t_{i+1} > M$, so $v^{t+1}_{i+1} = M$, and we calculate $h^{t+2}_i = M + f - \tau$. That means the correspondence C^{t+2}_{i-1} has $h^{t+2}_i = M + f - \tau$ and hence the (iii) case.

(ii) $h^t_{i+1} < M + f - \tau$:
If $v^t_{i+2} = \tau$, C^t_i connects to its preceding properly. If $v^t_{i+2} > \tau$, we have $h^{t+1}_{i+1} > h^t_{i+1} + \tau - f$, and $h^t_{i+1} > h^t_{i+1} + \tau - f$ since $v^t_{i+1} > \tau$, and $M > h^t_{i+1} + \tau - f$ since $h^t_{i+1} < M + f - \tau$. So $v^{t+1}_{i+1} > h^t_{i+1} + \tau - f$ and we calculate $h^{t+2}_i > h^t_{i+1}$. That means the correspondences of h^t_{i+1} at time $t+2, t+4, t+6, \ldots$ (*i.e.* $h^{t+2}_i, h^{t+4}_{i-1}, h^{t+6}_{i-2}, \ldots$) strictly increase until $M+f-\tau$ or until the correspondent front connects to its preceding properly.

(iii) $h^t_{i+1} = M + f - \tau$:
First check that $h^t_{i+1} = M + f - \tau$ implies $h^{t+2}_i = M + f - \tau$. And if $v^{t-1}_{i+2} \neq M$, we can consider the correspondent front C^{t+2}_{i-1} with $v^{t+1}_{i+1} = M$. So it does not matter to assume $v^{t-1}_{i+2} = M$. Then $h^{t-1}_{i+2} \geq v^{t-1}_{i+2} = M$, so we have $v^t_{i+2} = \min(M, h^t_{i+2})$. Now consider the following subcases:

(a) $h^t_{i+2} > 2M - \tau$:
We have $h^{t+2}_{i+1} = 2M - \tau, v^{t+2}_{i+1} = M$.

(b) $M \leq h^t_{i+2} < 2M - \tau$:
If $v^t_{i+3} = \tau$, C^t_i connects to its preceding properly. If $v^t_{i+3} > \tau$, note that $v^t_{i+2} = M$ and do similar calculation as in (ii), we have $h^{t+2}_{i+1} > h^t_{i+2}$.

(c) $h^t_{i+2} < M$:
If $v^{t+1}_{i+2} = \tau$, we have $h^t_{i+2} = \tau$ or $h^{t+1}_{i+2} = \tau$. In the former case, C^t_i connects to its preceding properly. In the latter case, v^t_{i+3} must be τ since $v^t_{i+2} = h^t_{i+2}$ implies $h^{t+1}_{i+2} = v^t_{i+3}$. Then again C^t_i connects to its preceding properly. If $v^{t+1}_{i+2} > \tau$, we calculate $h^{t+2}_{i+1} = v^t_{i+2} - \tau + v^{t+1}_{i+2} > v^t_{i+2} = h^t_{i+2}$.

Anyway, we will finally find a correspondence C^{t+2n}_{i-n} such that $h^{t+2n}_{i+2-n} = 2M - \tau$, $v^{t+2n}_{i+2-n} = M$ if C^{t+2m}_{i-m} did not connect to its preceding properly for all $0 \leq m \leq n$. Next we consider the general case:
$v^t_k = M$, $h^t_k = 2M - \tau$ for all $i + 2 \leq k \leq j$ with a $j \geq i + 2$.
First check that this condition will be inherited by C^{t+2}_{i-1} as $v^{t+2}_{k-1} = M$, $h^{t+2}_{k-1} = 2M - \tau$ for all $i + 2 \leq k \leq j$. Similarly we can assume $v^{t-1}_{j+1} = M$ and have $v^t_{j+1} = \min(M, h^t_{j+1})$. Consider the following cases and calculate just as above:

(a) $h^t_{j+1} > 2M - \tau$:
We have $h^{t+2}_j = 2M - \tau$, $v^{t+2}_j = M$.

(b) $M \leq h^t_{j+1} < 2M - \tau$:
If $v^t_{j+2} = \tau$, C^t_i connects properly. If $v^t_{j+2} > \tau$, we have $h^{t+2}_j > h^t_{j+1}$.

(c) $h^t_{j+1} < M$:
If $v^{t+1}_{j+1} = \tau$, C^t_i connects properly. If $v^{t+1}_{j+1} > \tau$, we have $h^{t+2}_j > h^t_{j+1}$.

So if C_i^t has not connected properly to its preceding yet, we can then consider a correspondence C_{i-n}^{t+2n} such that $v_{k-n}^{t+2n} = M$, $h_{k-1}^{t+2n} = 2M - \tau$ for all $i+2 \le k \le j+1$.

Since the number of cars is limited, we conclude that C_i^t will finally connects properly to its preceding τ-front.

Using theorem 2.14, we can now calculate the flow-density relation at time $t \gg 0$. Obviously, a free state has a $Q = M\rho$ for $\rho \le \frac{1}{M+1}$ and a τ-uniform state has a $Q = \tau\rho$ for $\rho = \frac{1}{\tau+1}$. And we have calculated the flow-density relation of a τ-congested state as $Q = \frac{\tau-1}{2}\rho + \frac{1}{2}$ for $\frac{1}{2M-\tau+1} < \rho < \frac{1}{\tau+1}$. This is the result we have shown in Fig.1.

3. Stabilities and an $\mathbf{O(N^2)}$ algorithm

The next two lemmas show a remarkable character of model (1).

Lemma 3.1. *Assume $v_i^{t-1} \ge \alpha$. Then $h_i^t \le 2M - \alpha$ implies $h_i^{t+1} \le 2M - \alpha$.*

Proof. Consider the three different cases of v_i^t. (i) $v_i^t = h_i^t$. Then $h_i^{t+1} = h_i^t + v_{i+1}^t - v_i^t = v_{i+1}^t \le M$. (ii) $v_i^t = M$. Then $h_i^{t+1} = h_i^t + v_{i+1}^t - M \le h_i^t \le 2M - \alpha$. (iii) $v_i^t = h_i^{t-1}$. We have $h_i^t = h_i^{t-1} + v_{i+1}^{t-1} - v_i^{t-1} \le h_i^{t-1} + M - \alpha = v_i^t + M - \alpha$ by assumption. So $h_i^{t+1} \le h_i^t + M - v_i^t \le 2M - \alpha$.

Lemma 3.2. *Suppose we have a type A α-front C_i^t and assume that $v_{i-1}^{t-1} \ge \alpha, v_{i+1}^{t+1} \ge \alpha$. Then the necessary and sufficient condition for C_{i-1}^{t+2} to be a type A α-front is $\alpha \le h_{i-1}^t \le 2M - \alpha$.*

Proof. Necessity: If $\alpha > h_{i-1}^t$, we have $v_{i-1}^{t+1} < \alpha$ so $h_{i-1}^{t+2} > h_{i-1}^{t+1}$ which means $h_{i-1}^{t+2} > v_{i-1}^{t+2}$ and hence C_{i-1}^{t+2} is not a type A front. If $2M - \alpha < h_{i-1}^t$, we have $h_{i-1}^{t+2} > \alpha$ since $v_{i-1}^t \le M, v_{i-1}^{t+1} \le M$. So C_{i-1}^{t+2} is not a type A α-front.

Sufficiency: First note that $v_{i+1}^{t+1} \ge \alpha$ implies $h_i^{t+2} \ge h_i^{t+1}$ and since $h_i^{t+1} = v_{i+1}^t > \alpha$ we get $v_i^{t+2} > \alpha$. Next we consider the following two situations and show in each case $h_{i-1}^{t+2} = v_{i-1}^{t+2} = \alpha$.

(i) $\alpha \le h_{i-1}^t \le M$.
This case, we have $v_{i-1}^t \ge \alpha$ since both $h_{i-1}^{t-1} \ge v_{i-1}^{t-1} \ge \alpha$ and $h_{i-1}^t \ge \alpha$. So we calculate $h_{i-1}^{t+1} \le h_{i-1}^t \le M$ which implies $v_{i-1}^{t+1} = h_{i-1}^{t+1}$ and $h_{i-1}^{t+2} = v_{i-1}^{t+2} = \alpha$.

(ii) $M \le h_{i-1}^t \le 2M - \alpha$.
If $v_{i-1}^t = M$ we have $h_{i-1}^{t+1} \le M$. Or if $v_{i-1}^t = h_{i-1}^{t-1}$, then $h_{i-1}^t = h_{i-1}^{t-1} + v_i^{t-1} - v_{i-1}^{t-1} \le h_{i-1}^{t-1} + M - \alpha = v_{i-1}^t + M - \alpha$, so again $h_{i-1}^{t+1} = h_{i-1}^t + \alpha - v_{i-1}^t \le M$. Since $h_{i-1}^t \ge M$, we have $v_{i-1}^{t+1} = h_{i-1}^{t+1}$ and $h_{i-1}^{t+2} = v_{i-1}^{t+2} = \alpha$.

Combining lemma 3.1 with lemma 3.2 we get the next lemma.

Lemma 3.3. *Suppose C_i^{t+k} to be a type A α-front with $k \geq 1$. Assume that $v_{i+1}^{t+k+1} \geq \alpha$ and $v_{i-1}^{t+s} \geq \alpha$ for all $0 \leq s \leq k$. If there exists $1 \leq r \leq k$ such that $h_{i-1}^{t+r} \leq 2M - \alpha$, then C_{i-1}^{t+k+2} is a type A α-front.*

The proof is obvious.

Lemma 3.4. *Suppose C_i^{t+k} to be a type A α-front with $k \geq 0$. Assume that $v_{i+1}^{t+k+1} \geq \alpha$ and $v_{i-1}^{t+s} \geq \alpha$ for all $0 \leq s \leq k$. Then the necessary and sufficient condition for C_{i-1}^{t+k+2} to be a type A α-front is that $x_i^{t+k} - x_{i-1}^t - 1 \leq v_{i-1}^t + (k+1)M - \alpha$.*

Proof. Necessity: It is easy to check that if C_{i-1}^{t+k+2} is a type A α-front then x_{i-1}^{t+k+2} must be $x_i^{t+k} + \alpha - 1$. On the other hand we have $x_{i-1}^{t+k+2} = x_{i-1}^t + v_{i-1}^t + \sum_{j=1}^{k+1} v_{i-1}^{t+j} \leq x_{i-1}^t + v_{i-1}^t + (k+1)M$ so $x_i^{t+k} - x_{i-1}^t - 1 \leq v_{i-1}^t + (k+1)M - \alpha$.

Sufficiency: In the case $k = 0$, the assumption is $v_{i+1}^{t+1} \geq \alpha, h_{i-1}^t \leq v_{i-1}^t + M - \alpha, v_{i-1}^t \geq \alpha$. First note that $v_{i+1}^{t+1} \geq \alpha$ implies $v_i^{t+k+2} > \alpha$. If $h_{i-1}^t \leq M$, we have $h_{i-1}^{t+1} \leq h_{i-1}^t$ since $v_{i-1}^t \geq \alpha = v_i^t$, then $v_{i-1}^{t+1} = h_{i-1}^{t+1}$. If $h_{i-1}^t \geq M$, we have $h_{i-1}^{t+1} \leq M$ since $h_{i-1}^t \leq v_{i-1}^t + M - \alpha$, then again $v_{i-1}^{t+1} = h_{i-1}^{t+1}$. Anyway, $v_{i-1}^{t+1} = h_{i-1}^{t+1}$ implies $h_{i-1}^{t+2} = v_{i-1}^{t+2} = \alpha$.
In the case $k \geq 1$, assume C_{i-1}^{t+k+2} is not a type A α-front. Then by lemma 3.3 we have $h_{i-1}^{t+r} > 2M - \alpha$ for all $1 \leq r \leq k$. In particular, we have $h_{i-1}^t - \alpha + M \geq h_{i-1}^t - v_{i-1}^t + v_i^t = h_{i-1}^{t+1} > 2M - \alpha$ so $h_{i-1}^t > M$, then $v_{i-1}^{t+r} = M$ for all $1 \leq r \leq k$. That means $x_i^{t+k} - 1 = x_{i-1}^{t+k} + h_{i-1}^{t+k} > (x_{i-1}^t + v_{i-1}^t + (k-1)M) + (2M - \alpha)$ which leads to a contradiction.

Remark3.5. In fact lemma 3.4 is true even when $k = -1$. For this case, the expression $x_i^{t+k} - x_{i-1}^t - 1 \leq v_{i-1}^t + (k+1)M - \alpha$ becomes $h_{i-1}^t \leq v_{i-1}^t$, on the other hand we have $h_{i-1}^t \geq v_{i-1}^t$ and $h_{i-1}^t \geq v_i^{t-1} = v_i^t = \alpha$, so it is equivalent to $h_{i-1}^t = v_{i-1}^t \geq \alpha$, which is the necessary and sufficient condition for C_{i-1}^{t+1} to be a type A α-front.

Furthermore, if C_{i-1}^{t+k+2} is *not* a type A α-front, the proof of lemma 3.4 actually shows that $v_{i-1}^{t+r} = M$ for all $1 \leq r \leq k+1$. So this case we have $x_{i-1}^{t+k+2} = x_{i-1}^t + v_{i-1}^t + (k+1)M$ for $k \geq -1$, and $v_{i-1}^{t+k+2} = \min(M, h_{i-1}^{t+k+2})$ for $k \geq 0$.

Definition3.6. We say a type A α-front C_i^t **remains to** C_{i-k} if for all $0 \leq j \leq k$, C_{i-j}^{t+2j} is a type A α-front.

If we somehow know the information about a type A α-front C_i^{t+k} at a future time $t+k$, then lemma 3.4 provides a way for us to predict whether the front remains or not through the information at the current time t. If the front remains to C_{i-1}, we get the information about C_{i-1}^{t+k+2}; Even not, we also have h_{i-1}^{t+k+2} and v_{i-1}^{t+k+2} by remark 3.5. This will be actively used in section 4 to investigate the detailed behavior of a front. Here we only pick up an outstanding specialization.

Corollary 3.7. *Suppose C_i^t to be a type A τ_t-front. Then C_i^t remains to C_{i-k} precisely when $x_i^t - x_{i-j}^t \leq j(2M - \tau_t + 1) + v_{i-j}^t - M$ for all $0 \leq j \leq k$.*

Proof. Since we are thinking about a τ_t-front, lemma 3.4 can be freely used without care about the estimate of velocities. Note that if C_i^t remains to C_{i-j}, then we have $x_{i-j}^{t+2j} = x_i^t + j(\tau_t - 1)$, and C_i^t remains to C_{i-j-1} precisely when $x_{i-j}^{t+2j} - x_{i-j-1}^t - 1 \leq v_{i-1}^t + (2j+1)M - \tau_t$ by lemma 3.4. That is the expression $x_i^t + j(\tau_t - 1) - x_{i-j-1}^t - 1 \leq v_{i-1}^t + (2j+1)M - \tau_t$ or $x_i^t - x_{i-j-1}^t \leq (j+1)(2M - \tau_t + 1) + v_{i-j-1}^t - M$.

Roughly speaking, this corollary means that we cannot untie an l-car-long τ-jam unless there is an $l(2M - \tau + 1)$ gap. In particular, the smaller τ is, the more difficult we untie the jam. Also, even one car with headway τ can propagate a jam in not so strict conditions. This can somehow be broken if we introduce a driver's perspective, but random generated, rather uniformly distributed cars will almost always cluster behind the car with the narrowest headway.[14]

Theorem 3.8. *If $\tau_{t+2N} = \tau_t$, then $\tau_\infty = \tau_t$.*

Proof. If it is a free state or a uniform state at time $t + 2N$, the statement is certainly true. So we assume $G_{t+2N} \neq \emptyset$. Let $\tau = \tau_t$, take a τ-front C_k^{t+2N}. If C_k^{t+2N} is type B, since $\tau_{t+2N} = \tau_{t+2N-1}$, $\phi_{t+2N}(C_k^{t+2N})$ is a type A τ-front by lemma 2.6. So it does not matter for us to change the assumption to: $\tau_{t+2N-2} = \tau_t$ and there is a type A τ-front in G_{t+2N-2}. Under this assumption, take a type A τ-front C_k^{t+2N-2}, let $C_i^t = \phi_{t+1} \circ \ldots \circ \phi_{t+2N-3} \circ \phi_{t+2N-2}(C_k^{t+2N-2})$. By lemma 2.6, C_i^t must be a type A τ-front, and C_i^t remains to $C_k = C_{i-N+1}$. Then by corollary 3.7, we have $x_i^t - x_{i-j}^t \leq j(2M - \tau + 1) + v_{i-j}^t - M$ for all $0 \leq j \leq N - 1$. In particular, $x_i^t - x_{i-N+1}^t \leq (N-1)(2M - \tau + 1)$. By the periodic boundary condition, C_{i-N}^t is a type A τ-front identical with C_i^t, so we have $x_{i-N+1}^t = x_{i-N}^t + \tau = x_i^t - L + \tau$. This implies $L \leq N(2M - \tau + 1)$. Now take an arbitrary integer $pN + q \geq 0$ with $0 \leq q \leq N - 1$, use the periodic boundary condition we get $x_{i-pN}^t - x_{i-pN-q}^t \leq q(2M - \tau + 1) + v_{i-pN-q}^t - M$ so $x_i^t - x_{i-pN-q}^t \leq pL + q(2M - \tau + 1) + v_{i-pN-q}^t - M \leq (pN + q)(2M - \tau + 1) + v_{i-pN-q}^t - M$ hence by corollary 3.7 C_i^t remains eternally.

Theorem 3.8 provides a way to calculate τ_∞ within limited steps: τ_t will not change further if it did not increase within $2N$ steps, and $\tau_t \leq M$. So we simply calculate $2MN$ time steps then τ_{2MN} must be τ_∞. Every time step has a computational requirement in proportion to N, so this algorithm has a computational requirement in proportion to N^2. However, this method is far from elegant and depends on the periodic boundary condition. In next section we will develop a smarter and more precise way to predict the behavior of fronts.

4. Minimal fronts and strictly minimal fronts

Definition4.1. An α-front C_i^t being **minimal** means that $h_i^s \geq \alpha$ for any $s > t$. If the inequality holds strictly, we say the front is **strictly minimal**.

Evidently, a τ_t-front is always minimal.

Lemma 4.2. *If an α-front C_i^t is minimal (resp. strictly minimal), then $\phi_t(C_i^t)$ is minimal (resp. strictly minimal). Or more generally, let $C_j^{t-1} = \phi_t(C_i^t)$, then $h_j^s \geq \alpha$ for any $s \geq t$ if C_i^t is minimal; the inequality strictly holds if C_i^t is strictly minimal.*

Proof. If C_i^t is type B, this is obvious because $\phi_t(C_i^t) = C_i^{t-1}$. Now assume C_i^t to be type A, so $\phi_t(C_i^t) = C_{i+1}^{t-1}$. We prove that if there exists an $s \geq t$ such that $h_{i+1}^s = \delta \leq \alpha$, then $h_i^{s+2} \leq \delta$ (this implies $\delta = \alpha$ when C_i^t is minimal, and leads to a contradiction when C_i^t is strictly minimal):
Since C_i^t is minimal, we have $h_i^r \geq \alpha$ for all $r \geq t$, so $v_i^r \geq \alpha$ for all $r \geq t$. And C_i^t is a type A front so $h_i^{t+1} = v_{i+1}^t \leq M \leq 2M - \alpha$. Then by lemma 3.1, we have $h_i^r \leq 2M - \alpha$ for all $r \geq t$. In particular, $h_i^s \leq 2M - \alpha$. Now do the following case division which is similar to the one in lemma 3.2:

(i) If $\alpha \leq h_i^s \leq M$:
Since $v_i^s \geq \alpha$ we have $h_i^{s+1} \leq h_i^s + h_{i+1}^s - \alpha = h_i^s + \delta - \alpha \leq h_i^s$, so $v_i^{s+1} = h_i^{s+1}$, then $h_i^{s+2} = v_{i+1}^{s+1} \leq h_{i+1}^s = \delta$.

(ii) If $M \leq h_i^s \leq 2M - \alpha$:
If $v_i^s = M$ we have $h_i^{s+1} \leq M$. Or if $v_i^s = h_i^{s-1}$, then $h_i^s = h_i^{s-1} + v_{i+1}^{s-1} - v_i^{s-1} \leq h_i^{s-1} + M - \alpha = v_i^s + M - \alpha$, so again $h_i^{s+1} = h_i^s + \alpha - v_i^s \leq M$. Since $h_i^s \geq M$, we have $v_i^{s+1} = h_i^{s+1}$ and hence $h_i^{s+2} = v_{i+1}^{s+1} \leq h_{i+1}^s = \delta$.

Corollary 4.3. *$\tau_t = \tau_\infty$ if and only if there is no strictly minimal τ_t-front at time t.*

Proof. Necessity (Need the periodic boundary condition): If there is a strictly minimal τ_t-front C_i^t, we have $h_i^s > \tau_t$ for all $s \geq t+1$. Using (ii) of lemma 2.2, we get $h_{i-1}^s > \tau_t$ for all $s \geq t+3$. Do this repeatedly and note the periodic boundary condition, finally we get $h_i^s > \tau_t$ for all $s \geq t + 2N - 1$ and all i. That means $\tau_\infty > \tau_t$.

Sufficiency: If $\tau_\infty > \tau_t$, there exists an $s \geq t$ such that $\tau_{s+1} > \tau_s$. Then all τ_s-fronts in G_s are strictly minimal. Take a $C_i^s \in G_s$, the front $\phi_{t+1} \circ \ldots \circ \phi_{s-1} \circ \phi_s(C_i^s) \in F_t$ is strictly minimal by lemma 4.2.

Notation4.4. Let S_t be the set consists of all minimal fronts at time t. Use Φ_t^{t-k} to denote the map $\phi_{t-k+1} \circ \ldots \circ \phi_{t-1} \circ \phi_t|_{S_t} : S_t \to S_{t-k}$. Φ_t^t is understood as the identity map of S_t.

Lemma 4.5. *Suppose C_i^t to be a type A minimal α-front. Then we have:*

1. *$v_{i+1}^s \geq \alpha$ for any $s \geq t$. If C_i^t is strictly minimal, the inequality holds strictly.*
2. *If C_i^t remains to C_{i-1}, then C_{i-1}^{t+2} is minimal. If C_i^t is strictly minimal, C_{i-1}^{t+2} is also strictly minimal.*
3. *$v_i^{t+2} = h_i^{t+1} = v_{i+1}^t$.*

Proof. (1) Since C_i^t is type A, by lemma 3.4 we have $h_{i+1}^s \geq \alpha$ for any $s \geq t$. So $v_{i+1}^s \geq \alpha$ holds for any $s \geq t$. (2) Since $h_i^s \geq \alpha$ for any $s \geq t+1$, using (ii) of lemma 2.2 we get $h_{i-1}^s \geq \alpha$ for any $s \geq t+3$. (3) Note that $v_{i+1}^{t+1} \geq \alpha$ and this can be checked by simple calculation.

Using lemma 4.5, we find that the condition $v_{i+1}^{t+k+1} \geq \alpha$ in lemma 3.4 is always satisfied if we assume C_i^{t+k} to be minimal. Then using (iii) of lemma 2.2 to estimate the velocities, we get the following generalization of corollary 3.7.

Corollary 4.6. *Suppose C_i^t to be a type A minimal α-front and assume $v_{i-j}^t \geq \alpha$ for all $0 \leq j \leq k$. Then C_i^t remains to C_{i-k} precisely when $x_i^t - x_{i-j}^t \leq j(2M - \tau_t + 1) + v_{i-j}^t - M$ for all $0 \leq j \leq k$.*

Proof. Since C_i^t is minimal, we have $v_i^s \geq \alpha$ for all $s \geq t$. The assumption is $v_{i-j}^t \geq \alpha$ for all $0 \leq j \leq k$. Using (iii) of lemma 2.2 repeatedly we conclude $v_{i-j}^s \geq \alpha$ for all $0 \leq j \leq k$ and all $s \geq t$. So the premise of lemma 3.4 is satisfied.

Now what happens if a front does not remain? Suppose $C_i^{t+k}(k \geq 0)$ to be a type A minimal α-front which does *not* remain to C_{i-1}. If we have $v_{i-1}^s \geq \alpha$ for all $t \leq s \leq t+k$, using remark 3.5 we can get h_{i-1}^{t+k+2} and $v_{i-1}^{t+k+2} = \min(M, h_{i-1}^{t+k+2})$. If $\beta = h_{i-1}^{t+k+2} < v_i^{t+k+2}$, we find that C_{i-1}^{t+k+2} is a type A β-front. If not the case, generally we cannot say more without some extra information about the cars before C_{i+1}^{t+k}. However, we will show that as for strictly minimal fronts, a prediction indeed can be done since the situation becomes simple enough after a specific time point.

Lemma 4.7. *Suppose C_i^t to be a type A minimal α-front. Then $v_{i+1}^{t+1} \leq v_i^{t+3}$. If C_i^t is strictly minimal, the inequality strictly holds unless $v_{i+1}^{t+1} = v_i^{t+3} = M$.*

Proof. By lemma 4.5, we have $v_i^{t+2} = h_i^{t+1} = v_{i+1}^t > \alpha$ and $v_{i+1}^{t+2} \geq \alpha$, so $h_i^{t+2} = h_i^{t+1} - \alpha + v_{i+1}^{t+1} > v_{i+1}^{t+1}$, and $h_i^{t+3} = (h_i^{t+1} - v_i^{t+2}) + (v_{i+1}^{t+2} - \alpha) + v_{i+1}^{t+1} \geq v_{i+1}^{t+1}$. Hence $v_{i+1}^{t+1} \leq v_i^{t+3}$.

Lemma 4.8. *At time $t \geq 2M^2$, a strictly minimal front C_i^t must be in one of the following two cases.*

1. *A type A front with $v_{i+1}^t < M, v_{i+1}^{t+1} = M$;*
   Or a type B front with $h_i^t < M, v_{i+1}^t = M, v_i^{t+2} = M$.
2. *A type A front with $v_{i+1}^t = M$;*
   Or a type B front with $v_i^{t+1} = M$.

Proof. First we prove that if $C_j^s = \phi_{s+1}(C_k^{s+1})$ and C_k^{s+1} is a strictly minimal front, then C_j^s in case (1) or (2) implies C_k^{s+1} in case (1) or (2).

- C_j^s is a type A α-front in case (1):
  Since C_j^s is type A, $C_k^{s+1} = C_j^{s+1}$ is type B and $h_k^{s+1} = v_{j+1}^s < M, v_{k+1}^{s+1} = v_{j+1}^{s+1} = M$ immediately follows. We have $h_k^{s+2} \geq v_{k+1}^{s+1} = M$, and apply

lemma 4.5 to the type A minimal α-front C_j^s we get $v_{k+1}^{s+2} \geq \alpha$, so $h_k^{s+3} = h_k^{s+1} - \alpha - v_k^{s+2} + M + v_{k+1}^{s+2} = M + v_{k+1}^{s+2} - \alpha \geq M$, that means $v_k^{s+3} = M$. Hence C_k^{s+1} is in case (1).

- C_j^s is type B in case (1):
If $j = k$ and C_k^{s+1} is type B, then $v_k^{s+2} = v_j^{s+2} = M$, so C_k^{s+1} is in case (2). If $k = j-1$ and C_k^{s+1} is type A, we have $v_{k+1}^{s+1} = v_j^{s+1} = h_j^s < M$, and $v_{k+1}^{s+2} = v_j^{s+2} = M$, so C_k^{s+1} is in case (1).
- C_j^s is type A in case (2):
Then $C_k^{s+1} = C_j^{s+1}$ is type B and we have $v_k^{s+2} = v_j^{s+2} = v_{j+1}^s = M$ by lemma 4.5. So C_k^{s+1} is in case (2).
- C_j^s is type B in case (2):
$v_j^{s+1} = M$ so C_j^{s+1} cannot be a front. Hence $k = j-1$ and C_k^{s+1} is type A, with $v_{k+1}^{s+1} = v_j^{s+1} = M$. So C_k^{s+1} is in case (2).

Let $K_r = \Phi_t^{t-2M^2+r}(C_i^t)$. Consider the sequence $K_1, K_2, \ldots, K_{2M^2}$. If we can find a K_r in case (1) or (2), then C_i^t must be in case (1) or (2). Insert a dividing line between K_p and K_{p+1} if they are both type B. The total number of dividing lines is less than $(M-1)$, because if we have K_q as an α-front and K_{q+1} as a β-front, then basically $\beta \geq \alpha$, and the inequality is strict if K_q and K_{q+1} are both type B, which means this both type B case cannot happen more than $(M-1)$ times. Now these less than $(M-1)$ dividing lines divide the sequence into less than M parts, and since the sequence is $2M^2$ long, we can find a part whose length exceeds $2M$. Every type A front in this part is followed by a type B front and vice versa, so we finally found a $2M$ long sequence $K_a, K_{a+1}, \ldots, K_{a+2M-1}$ with an A,B,A,B,... type pattern. Then apply lemma 4.7 to $K_a = C_x^y, K_{a+2} = C_{x-1}^{y+2}, K_{a+4} = C_{x-2}^{y+4}, \ldots$ we found that $v_{x+1}^{y+1}, v_x^{y+3}, v_{x-1}^{y+5}, \ldots$ increases to M at last. Then the type A front $K_{a+2M-2} = C_{x-M+1}^{y+2M-2}$ with $v_{x-M+2}^{y+2M-1} = M$ is in case (1) if $v_{x-M+2}^{y+2M-2} < M$, or in case (2) if $v_{x-M+2}^{y+2M-2} = M$.

Remark4.9. Now suppose $t \geq 2M^2$, and C_i^t is a type A strictly minimal α-front which does not remain to C_{i-1}. Assume we know h_{i-1}^{t+2} and $v_{i-1}^{t+2} = \min(M, h_{i-1}^{t+2})$. Then lemma 4.8 suggests all cases would happen:

(i) C_i^t is in case (1). Let $f = h_i^t$.
  (a) If $\beta = h_{i-1}^{t+2} < f$, then C_{i-1}^{t+2} is a type A β-front, $v_i^{t+2} = f, v_i^{t+3} = M$.
  (b) If $f \leq h_{i-1}^{t+2} \leq M$, then $x_{i-1}^{t+3} = x_i^{t+2} - 1$, C_{i-1}^{t+3} is a type A f-front, $v_i^{t+3} = M$.
  (c) If $M < h_{i-1}^{t+2} < 2M - f$, then $x_{i-1}^{t+3} = x_{i-1}^{t+2} + M$, C_{i-1}^{t+3} is a type A γ-front where $\gamma = h_{i-1}^{t+2} - M + f$, $v_i^{t+3} = M$.
  (d) If $h_{i-1}^{t+2} \geq 2M - f$, then $v_{i-1}^{t+2} = v_{i-1}^{t+3} = v_i^{t+3} = M$.

(ii) C_i^t is in case (2).
  (a) If $\beta = h_{i-1}^{t+2} < M$, then C_{i-1}^{t+2} is a type A β-front, $v_i^{t+2} = M$.

(b) If $h_{i-1}^{t+2} \geq M$, then $v_{i-1}^{t+2} = v_i^{t+2} = M$.

Readers are perhaps worried about how do we know a front remains or not when the estimates of velocities do *not* hold? In fact this case is avoided well in our algorithm. The mechanism is based on the following lemma, which suggests that if we check from 0-fronts to $(M-1)$-fronts in turn, things will go smoothly.

Lemma 4.10. *Suppose $C_i^{t+k}(k \geq 0)$ to be a type A minimal α-front. If C_{i-1}^t is NOT a β-front with $\beta < \alpha$, and C_{i-1}^s is NOT a type A strictly minimal γ-front for any $s \geq t+1$ and any $\gamma < \alpha$, then $v_{i-1}^s \geq \alpha$ for all $s \geq t$.*

Proof. Let $\delta = \min\{h_{i-1}^s | s \geq t\}$. Since C_i^{t+k} is minimal we have $h_{i-1}^s \geq \alpha$ for all $s \geq t+k+2$ by (ii) of lemma 2.2, so if $\delta < \alpha$, the minimum δ is gained at a time before $t+k+2$. Let $r = \max\{s \geq t | h_{i-1}^s = \delta\}$ and we find C_{i-1}^r to be a type A strictly minimal δ-front if $r \geq t+1$. As this contradicts our assumption, r must be t. This case, since C_{i-1}^t is not a β-front with $\beta < \alpha$, we have $v_i^t \leq v_{i-1}^t = h_{i-1}^t = \delta$, but this implies $h_{i-1}^{t+1} \leq \delta$ and contradicts the definition of r. So we conclude that $h_{i-1}^s \geq \alpha$ for all $s \geq t$ and hence $v_{i-1}^s \geq \alpha$ for all $s \geq t+1$. Furthermore, $v_{i-1}^t \geq \alpha$ holds because if $\epsilon = v_{i-1}^t < \alpha$ then C_{i-1}^t is a type B ϵ-front, contradiction again.

Remark4.11. It is valuable to point out a special case.

First we make an understanding that for a type B α-front C_x^y and a $z \geq 1$, we say C_x^y remains to C_{x-z} in the meaning that C_{x-1}^{y+1} is a type A α-front which remains to C_{x-z}.

Now let $t \geq 2M^2$, let $C_i^{t+k}(k \geq 0)$ be a type A minimal α-front and C_{i-1}^t a type B strictly minimal β-front with $\beta < \alpha$. Suppose C_{i-1}^t remains to C_{i-p}, C_{i-1}^s is *not* a type A strictly minimal γ-front for any $s \geq t+1$ and any $\gamma < \alpha$. We want to know whether C_i^{t+k} remains to C_{i-q} for any $1 \leq q \leq p$?

First we consider C_{i-1}. Lemma 4.8 says there are two cases:

- C_{i-1}^t satisfies (1) of lemma 4.8.
  We have $v_{i-1}^{t+2} = M$. So C_{i-1}^{t+2} is not a front. Using lemma 4.10 we get $v_{i-1}^s \geq \alpha$ for all $s \geq t+2$. Note that k cannot be 0 because $v_i^t = M$. For $k \geq 1$, using lemma 3.4 we conclude that C_i^{t+k} remains to C_{i-1} precisely when $x_i^{t+k} - x_{i-1}^{t+2} - 1 \leq kM - \alpha$.

- C_{i-1}^t satisfies (2) of lemma 4.8.
  We have $v_{i-1}^{t+1} = M$, so C_{i-1}^{t+1} is not a front, then $v_{i-1}^s \geq \alpha$ for all $s \geq t+1$. Using lemma 3.4 we get the condition $x_i^{t+k} - x_{i-1}^{t+1} - 1 \leq (k+1)M - \alpha$.

These two expressions can be unified to $x_i^{t+k} - x_{i-1}^{t+1} - 1 \leq v_{i-1}^{t+1} + kM - \alpha$.

Now we consider C_{i-q}. Note that the condition $v_{i-1}^s \geq \alpha$ for all $s \geq t+2$ (or $s \geq t+1$) is equivalent to $h_{i-1}^s \geq \alpha$ for all $s \geq t+1$ (or $s \geq t$), so we use (ii) of lemma 2.2 to conclude that the velocity estimates for C_{i-q} automatically hold. If C_{i-1}^t satisfies (1) of lemma 4.8, we have $v_{i-q}^{t+2q-1} = v_{i-1}^{t+1}$ by lemma 4.5, and $v_{i-q}^{t+2q} = M$ by lemma 4.7; if C_{i-1}^t satisfies (2) of lemma 4.8, we have $v_{i-q}^{t+2q-1} = M$

by lemma 4.5. Anyway, the calculation is the same to the C_{i-1} case. Note that $x_{i-q}^{t+2q-1} = x_{i-1}^{t+1} + (q-1)(\beta - 1)$, and $x_{i-q+1}^{t+k+2q-2} = x_i^{t+k} + (q-1)(\alpha - 1)$ if C_i^{t+k} remains to C_{i-q+1}, we get the condition for C_i^{t+k} to remain to C_{i-q} as: $x_i^{t+k} - x_{i-1}^{t+1} - 1 \leq v_{i-1}^{t+1} + kM - \alpha - (q-1)(\alpha - \beta)$.

This expression shows that it is automatically determined by the index q whether C_i^{t+k} remains to C_{i-q}. Its analogue to the $\beta = \alpha$ case suggests the following.

Lemma 4.12. *Suppose C_i^{t+k} to be a type A minimal α-front and C_{i-1}^t a type B minimal α-front. If C_{i-1}^t remains to C_{i-p}, then C_i^{t+k} remains to C_{i-p} precisely when C_i^{t+k} remains to C_{i-1}.*

Proof. This can be easily showed by lemma 3.3. Note that the estimate of velocities automatically holds, and if C_{i-1}^t remains to C_{i-2}, then $h_{i-2}^{t+1} = \alpha \leq 2M - \alpha$.

The next lemma is a crucial application of lemma 4.8 which essentially makes the algorithm work.

Lemma 4.13. *Let $t \geq 2M^2$. If $C_i^{t+k}, C_j^{t+k} \in S_{t+k}$ and $i < j$, then we have $\Phi_{t+k}^t(C_i^{t+k}) \neq \Phi_{t+k}^t(C_j^{t+k})$ unless the case $j = i+1$ and $\phi_{t+k}(C_i^{t+k}) = \phi_{t+k}(C_{i+1}^{t+k})$. More precisely, if a type A front C_p^s and a type B front C_{p+1}^s at time $s \geq 2M^2$ are both minimal, then there does NOT exist a front C_q^{s+1} such that $\phi_{s+1}(C_q^{s+1}) = C_{p+1}^s$.*

Proof. Let C_p^s be an α-front and C_{p+1}^s a β-front. Then we have $\beta > \alpha$ by definition. Since C_{p+1}^s is minimal, $h_{p+1}^r \geq \beta > \alpha$ for all $r \geq s$. Using (ii) of lemma 2.2 we get $h_p^r > \alpha$ for all $r \geq s+2$. $h_p^{s+1} > \alpha$ also holds because C_p^s is type A. Hence C_p^s is a strictly minimal front which must be in case (1) or (2) of lemma 4.8. $v_{p+1}^{s+1} < M$ because C_{p+1}^s is a front, so it is case (1). Then we find C_p^{s+1} to be a type B front with $v_{p+1}^{s+1} = M$. That means there is no front can be mapped to C_{p+1}^s.

According to this lemma, if we restrict Φ_{t+k}^t to G_{t+k} or to the set consists of all type A minimal fronts at time $t+k$, then we get a one-to-one map. Moreover, the reverse of this map can be searched like this:
For minimal front C_i^t, if C_i^t is type A, then we next go to C_i^{t+1}; if C_i^t is type B, then we go to C_{i-1}^{t+1} when C_{i-1}^{t+1} is a type A front, or to C_i^{t+1} when not.

Notation4.14. Notation for us to describe the algorithm.

- If $C_j^s = \Phi_t^s(C_i^t)$, we say C_i^t is **mapped to** C_j^s, or C_j^s is an **image** of C_i^t.
- For a minimal front C_i^t, we say $P(C_i^t)$ is the **domain** of C_i^t and define it as $P(C_i^t) = \{C_j^{s-k} | C_j^s = \Phi_t^s(C_i^t), 0 \leq k \leq s\}$. Give a partial order "$\prec$" between two minimal fronts as $C_i^t \prec C_j^s \Leftrightarrow P(C_i^t) \subseteq P(C_j^s)$. Note that $C_i^t \prec C_j^s$ is equivalent to $C_i^t \in P(C_j^s)$.

- If a type A minimal front C_i^t remains to C_{i-k} but does not remain to C_{i-k-1}, we define $d(C_i^t)$ to be C_{i-k}^{t+2k+1}. Similarly, if a type B minimal front C_i^t remains (remark 4.11) to C_{i-k} but does not remain to C_{i-k-1}, we define $d(C_i^t)$ to be C_{i-k}^{t+2k}. Anyway, $d(C_i^t)$ is a type B minimal front and called the **destination** of C_i^t. If C_i^t remains eternally, we say $d(C_i^t) = \infty$.

- For a type B strictly minimal front C_i^t which does not remain to C_{i-1}, let $e(C_i^t) = C_{i-1}^{t+1}$ if C_{i-1}^{t+1} is a type A front, and $e(C_i^t) = C_{i-1}^{t+2}$ if C_{i-1}^{t+1} is *not* a type A front but C_{i-1}^{t+2} *is*. In other case, $e(C_i^t)$ is nothing. (*cf.* remark 4.9)

- Let X be the set of all cars at all time points and F the set of all fronts. *i.e.* $X = \{C_i^t | i \in \mathbf{Z}, t \geq 0\}$, $F = \bigcup_{t=0}^{\infty} F_t$. For $X' \subseteq X, F' \subseteq F$, we say F' is **complete** to X' if for any type A minimal front $C_i^t \in X'$, there exists a (minimal) front $C_j^s \in F'$ such that $C_j^s = \Phi_t^s(C_i^t)$.

Now we are ready to see the algorithm. We first show the main program and prove its correctness, then show the details of the subroutine.

Algorithm4.15. First of all, calculate $2M^2$ time steps and reset the time to 0.

- `let` $Y_0 = F_0$
- `if` $F_0 = \emptyset$ `then` it is a free or uniform state, end the program.
- `for` $\tau = 0$ `to` $M-1$
  - `let` $\Lambda_\tau = \{\tau\text{-fronts in } Y_\tau\}$.
  - `subroutine`:
    calculate $D_\tau = \{d(C_i^t) | C_i^t \in \Lambda_\tau\}$. If $\infty \in D_\tau$, output $\tau_\infty = \tau$ and $R = \{C_i^t \in \Lambda_\tau | d(C_i^t) = \infty\}$, end the program. If $\infty \notin D_\tau$, find the maximal elements of D_τ in the partial order "$\prec$", put these elements into Z_τ, and calculate $Y'_\tau = Y_\tau \setminus \bigcup_{C_i^t \in Z_\tau} P(C_i^t)$, $E_\tau = \{e(C_i^t) | C_i^t \in Z_\tau\}$.
  - `let` $Y_{\tau+1} = Y'_\tau \cup E_\tau$.
- conclude $\tau_\infty = M$, end the program.

Proof of correctness. Let $X_0 = X$ and $X_{\tau+1} = X_\tau \setminus \bigcup_{C_i^t \in Z_\tau} P(C_i^t)$ if $\infty \notin D_\tau$. Obviously the elements of Λ_0 are minimal, Y_0 is complete to X_0, and there is no α-front in Y_0 with $\alpha < 0$. By induction, we can assume that the elements of Λ_τ are minimal, Y_τ is complete to X_τ, and there is no α-front in Y_τ such that $\alpha < \tau$. If $\infty \in D_\tau$, of course $\tau_\infty = \tau$, and by the completeness of Y_τ, we can calculate all τ_∞-fronts at time $t \gg 0$ from R. Those τ_∞-fronts finally connect properly to their preceding fronts by theorem 2.14, so the final stationary state is then entirely understood. Now we consider the $\infty \notin D_\tau$ case.

(i) There is no type A minimal β-front with $\beta < \tau$ exists in X_τ:
If we have a type A minimal β-front in X_τ, it is mapped to an α-front with $\alpha \leq \beta$ in Y_τ since Y_τ is complete to X_τ. However, there is no α-front in Y_τ such that $\alpha < \tau$.

(ii) $\tau_\infty > \tau$:
Since $\infty \notin D_\tau$, we can select a time point $s > \max\{t | C_i^t \in Z_\tau\}$, then there is no type A minimal β-front with $\beta < \tau$ exists after time s by (i). We will show β cannot be τ either. That is because if such a β-front exists, it is mapped to a C_j^r in Λ_τ, and so C_j^r remains to a time after s. That contradicts the definition of s.

(iii) Elements of Z_τ are strictly minimal:
Take a $C_i^t \in Z_\tau$ and assume $\delta = \min\{h_i^s | s > t\} \leq \tau$. The elements of Λ_τ are minimal so by lemma 4.5 C_i^t is also minimal. Then $\delta = \tau$. Since $\tau_\infty > \tau$, we can take $r = \max\{s > t | h_i^s = \delta\}$ and find C_i^r to be a type A minimal τ-front. This contradicts the definition of Z_τ, for we have $C_i^t \prec C_i^r$, and C_i^r is mapped to a front C_j^u in Λ_τ, then $d(C_j^u) \succ C_i^r \succ C_i^t$.

(iv) For any $C_i^t \in Z_\tau$, let s be such that $C_{i-1}^s \in X_\tau$ and $C_{i-1}^{s-1} \notin X_\tau$, then $v_{i-1}^r = M$ for all $s+1 \leq r \leq t$ (*i.e.* the "left border" of $P(C_i^t)$ is coated by a "velocity M wall"):
If $t = 0$, there is nothing to prove. If $t > 0$, then C_i^{t-1} is a type A minimal τ-front. If $s = 0$, then $C_{i-1}^s \in Y_\tau$, so C_{i-1}^s is not an α-front with $\alpha < \tau$. Then by lemma 4.10 we have $v_{i-1}^r \geq \tau$ for all $r \geq s$, and since C_i^{t-1} does not remain to C_{i-1}, we have $v_{i-1}^r = M$ for all $s+1 \leq r \leq t$ by remark 3.5. If $s > 0$, C_{i-1}^{s-1} must be a type B strictly minimal β-front with $\beta < \tau$ by (iii). Since $C_i^{t-1} \in X_\tau$, we have $t-1 \geq s-1$ (the $t = s-1$ or $t = s-2$ case is avoided since C_{i-1}^{s-1} is strictly minimal and by lemma 4.8). So as we have discussed in remark 4.11, we also have the velocity estimates and then can use remark 3.5.

(v) $Y_{\tau+1}$ is complete to $X_{\tau+1}$:
Y_τ is complete to X_τ and so complete to $X_{\tau+1}$. Then for any minimal type A front $C_j^s \in X_{\tau+1}$, there exists a $C_k^r \in Y_\tau$ such that $C_k^r = \Phi_s^r(C_j^s)$. If $C_k^r \in Y_{\tau+1}$ there is nothing to prove. If $C_k^r \notin Y_{\tau+1}$, since $Y_{\tau+1} \supseteq Y_\tau \setminus \bigcup_{C_i^t \in Z_\tau} P(C_i^t)$, there exists a $C_i^t \in Z_\tau$ such that $C_k^r \in P(C_i^t)$. On the other hand $C_j^s \notin P(C_i^t)$, so we can find a $s \leq u \leq r$ such that $\Phi_s^u(C_j^s) \in P(C_i^t)$ and $\Phi_s^{u+1}(C_j^s) \notin P(C_i^t)$. Let C_l^u denote $\Phi_s^u(C_j^s)$. If $\Phi_s^{u+1}(C_j^s) = C_{l-1}^{u+1}$ (*i.e.* C_l^u is at the "left border" of P(C_i^t)), then $l = i$, so C_l^u must be C_i^t by (iv). If $\Phi_s^{u+1}(C_j^s) = C_l^{u+1}$ (*i.e.* C_l^u is at the "right border" of P(C_i^t)), then C_l^u is an image of C_i^t, so by lemma 4.13, again C_l^u must be C_i^t. Anyway C_i^t is an image of C_j^s, and now $e(C_i^t)$ is the first type A front mapped to C_i^t, so we conclude that $e(C_i^t)$ is an image of C_j^s.

(vi) There is no α-front with $\alpha < \tau + 1$ in $Y_{\tau+1}$:
Because $\Lambda_\tau \subseteq \bigcup_{C_i^t \in Z_\tau} P(C_i^t)$, so there is no τ-front in Y'_τ; and every element in E_τ is a type A γ-front with $\gamma > \tau$.

(vii) Elements of $\Lambda_{\tau+1}$ are minimal:
Take a $C_i^t \in \Lambda_{\tau+1}$ and assume $\delta = \min\{h_i^s | s > t\} \leq \tau$. Since $\tau_\infty > \tau$,

we can take $r = \max\{s > t | h_i^s = \delta\}$ and find C_i^r to be a type A minimal δ-front. Since $Y_{\tau+1}$ is complete to $X_{\tau+1}$, C_i^r is mapped to an ϵ-front with $\epsilon \leq \delta$ in $Y_{\tau+1}$. However there is no such an ϵ-front by (vi).

Now we have accomplished the induction and finished the proof.

In fact, any α-front lies in $P(C_i^t)(C_i^t \in Z_\tau)$ with $\alpha > \tau$ cannot be minimal, so this algorithm actually calculates the behavior of all minimal fronts.

Now we are going to see the details of the subroutine.

- `data structure`: The `SET` structure is a collection of C_i^ts, we can `delete` (resp. `add`) an element from (resp. into) the collection. Also, it has an index i, for a `SET` S, we can use the method S.`get`(i) to get an ith car C_i^t in S. If there is no ith car in S, the `get` method return `null`. The method S.`delete`(i) deletes all ith cars in S. Using "`for each` C_i^t $\in$S" we can enumerate all C_i^ts in S from small i to large i, when i is the same from small t to large t, in turn. Let Λ_τ and Y_τ have the `SET` structure.
- `initialization`: keep an array (x_i, v_i, t_i) with index i in the memory, initialize (x_i, v_i, t_i) to be $(x_i^0, v_i^0, 0)$ for all i.
- `subroutine begin`
- `let` i_{prev}=`null`,C_{prev}=`null`
- To avoid the technical difficulties due to the periodic boundary condition, here we normalize the index i of all $C_i^t \in \Lambda_\tau$ to $0 \leq i \leq N-1$. Then we double Λ_τ in the meaning that if $C_i^t \in \Lambda_\tau$ with $0 \leq i \leq N-1$, then $C_{i+N}^t \in \Lambda_\tau$.
- `for each` $C_i^t \in \Lambda_\tau$
  - `let` $(j, s, x) = (i, t, x_i^t)$
  - `if` $i = i_{prev}$, `then`:
    `if` $d(C_{prev}) = \infty$ `then` conclude $d(C_i^t) = \infty$, `next` C_i^t.
    `else`, we have $d(C_{prev}) = C_{i-p}^r$, `delete` C_{prev} from Λ_τ, `redefine` $(j, s, x) = (i-p, t+2p, x_i^t + p(\tau - 1))$. ($C_i^t$ remains to C_{i-p}, verified by lemma 4.12)
  - `redefine` $i_{prev} = i$, $C_{prev} = C_i^t$
  - `if` C_i^t is type B, `then`: (this case, t must be 0 by definition of Y_τ)
    `if` C_i^t does not remain to C_{i-1}, `then` conclude $d(C_i^t) = C_i^t$, `next` C_i^t.
    `else`, `redefine` $(j, s, x) = (i-1, t+1, x_i^t - 1)$
  - `while` $x - x_{j-1} - 1 \leq v_{j-1} + (s - t_{j-1} + 1)M - \tau$ `do`:
    (C_i^t remains to C_{j-1})
    - `If` Λ_τ.`get`$(j-1)$ is not `null` `then`:
      we have Λ_τ.`get`$(j-1)$=C_{j-1}^u,
      `if` $d(C_{j-1}^u) = \infty$, `then` conclude $d(C_i^t) = \infty$, `next` C_i^t.
      `else`, we have $d(C_{j-1}^u) = C_{j-q}^w$, `delete` C_{j-1}^u from Λ_τ, `redefine` $(j, s, x) = (j-q, s+2q, x - q(\tau - 1))$.

    * `else redefine` $(j, s, x) = (j - 1, s + 2, x + \tau - 1)$
    * `if` $j \le i - N$, `then` conclude $d(C_i^t) = \infty$, `next` C_i^t
  - conclude $d(C_i^t) = C_j^{s+1}$

- Now we can "undouble" Λ_τ by delete all $C_i^t \in \Lambda_\tau$ whose index $i \ge N$.

- D_τ is obtained. `if` $\infty \in D_\tau$, `then` output $\tau_\infty = \tau$ and $R = \Lambda_\tau$, end the program. `else`, $Z_\tau = \{d(C_i^t) | C_i^t \in \Lambda_\tau\}$.

- For every $C_i^t \in Z_\tau$, calculate $e(C_i^t)$. If $t = 0$, we simply check the data of time 0. If $t > 0$ then C_i^{t-1} is a type A strictly minimal τ-front, and we have done the calculation in remark 4.9.

- Refresh (x_i, v_i, t_i) and calculate Y'_τ: For every $C_i^t \in \Lambda_\tau$ with $d(C_i^t) = C_{i-p}^u$, if C_i^t is type A, `redefine` $(x_{i-q}, v_{i-q}, t_{i-q}) = (x_i^t + 2\tau + q(\tau - 1), v_i^{t+2}, t + 2 + 2q)$ and Y_τ`.delete(`$i - q$`)` for all $0 \le q \le p$; if C_i^t is type B, `redefine` $(x_{i-q}, v_{i-q}, t_{i-q}) = (x_i^t + \tau + q(\tau - 1), v_i^{t+1}, t + 1 + 2q)$ and Y_τ`.delete(`$i - q$`)` for all $0 \le q \le p$.

- The refresh of (x_i, v_i, t_i) is verified by the calculation in remark 4.11, and now the modified Y_τ is just Y'_τ.

- `subroutine end`

About the computational requirement of this algorithm, note that $\#E_\tau \le \#D_\tau = \#\Lambda_\tau$, so $\#Y_{\tau+1} \le \#Y_\tau \le \#Y_0 \le N$. One time the subroutine runs, the `while do` loop runs less than $4N$ times, because once we found C_i^t remains to C_{i-p}, the $[i - p, i - 1]$ interval is then skipped afterward. Similarly, the (x_i, v_i, t_i) refresh loop runs less than N times, because for every two C_i^t, C_j^s in (modified)Λ_τ with $d(C_i^t) = C_{i-p}^r, d(C_j^s) = C_{j-q}^u$, the interval $[i - p, i]$ and $[j - q, j]$ are disjoint. So we conclude that this is an $\mathrm{O}(N)$ algorithm. Besides, corollary 4.6 and remark 4.11 just can be used for optimizing the subroutine.

5. Summary and further discussion

We have classified all stationary states of Eq.(2), and proved that every state finally evolves to a stationary state if we assume the periodic boundary condition. The results about the "stability" of a τ-front show that the smaller τ is, the easier the front remains. Tools to investigate the detailed behavior of fronts are developed, briefly speaking, after a specific time point, strictly minimal fronts emerge in a clear shape and then dominate the time evolution of cars. We think these results grasped the main characters of model (1).

All these characters are described with the concept "front", which focus on the non-trivial behavior of cars in an accelerating process. We think this is an important concept for us to analyze traffic CA models in Lagrange form. For example, there is another widely used (such as in the NS model) effect which is considered to be associated with the inertia of cars. It requires that

$v_i^t \le v_i^{t-1}+1$, *i.e.* the acceleration cannot exceed 1. This effect however does not result in multiple branches in the fundamental diagram, and we find that if the term h_i^{t-1} of model (1) is replaced by $v_i^{t-1}+1$, the process for successive cars with headway τ to accelerate from velocity τ to the free speed M always leads to successive cars with headway M, but not the characteristic headway $2M-\tau$. We think analysis focused on such kind of behavior can reveal some nature of a traffic CA.

Now we are going to propose a generalization of model (1) which takes in a driver's anticipation. A kind of generalization is proposed by Nishinari,[12] but with too many terms to avoid a collision. Our generalized equation is written as:

$$u_i^t = \min(M, h_i^t, h_i^{t-1}) \tag{3}$$
$$v_i^t = \min(M, u_i^t + [\alpha_i u_{i+1}^t + \beta_i]) \tag{4}$$

h_i^t and v_i^t are as above. α_i and β_i are real numbers such that $0 \le \alpha_i \le 1, 0 \le \beta_i < 1$. $[\cdot]$ is known as the Gauss function, $[x]$ represents the maximal integer which does not exceed x. Let $\alpha_i = \beta_i = 0$ for all i we get Eq.(2).

Proposition 5.1. *There is no collision in this generalized model.*

PROOF. What to prove is that $h_i^{t+1} = h_i^t - v_i^t + v_{i+1}^t$ cannot be negative. If $v_{i+1}^t = M$, this is obvious. So we assume $v_{i+1}^t = u_{i+1}^t + [\alpha_{i+1}u_{i+2}^t + \beta_{i+1}] \ge u_{i+1}^t$. $h_i^t \ge u_i^t$ by Eq.(3), $v_i^t \le u_i^t + [\alpha_i u_{i+1}^t + \beta_i]$ by Eq.(4). So it is enough to prove $u_i^t - (u_i^t + [\alpha_i u_{i+1}^t + \beta_i]) + u_{i+1}^t \ge 0$. Note that $[\alpha_i u_{i+1}^t + \beta_i] \le [u_{i+1}^t + \beta_i] = u_{i+1}^t$, so the inequality follows.

Note that the parameter α_i and β_i can be chosen differently for each i, or vary randomly in the time evolution. The NS model introduced a random braking which is known to be responsible for spontaneous jam formation, however it is likely to find a spontaneous jam formation in this generalized model when we set each α_i and β_i differently to get a but deterministic model. When $\alpha_i = \alpha, \beta_i = \beta$ for all i, we have an analogue to lemma 2.2 as the following:

Proposition 5.2. *When $\alpha_i = \alpha, \beta_i = \beta$ for all i, the following inequalities hold:*

(i) $h_i^{t+1} \ge \min(h_i^t, u_{i+1}^t, u_{i+2}^t)$
(ii) $h_i^{t+1} \ge \min(M, h_i^t, h_{i+1}^t, h_{i+2}^t, h_{i+1}^{t-1}, h_{i+2}^{t-1})$
(iii) $u_i^{t+1} \ge \min(u_i^t, u_{i+1}^t, u_{i+2}^t)$

PROOF. (i) If $v_{i+1}^t = M$, then $h_i^{t+1} \ge h_i^t$. Now we assume $v_{i+1}^t = [\alpha u_{i+2}^t + \beta] + u_{i+1}^t$. Then

$$\begin{aligned}
h_i^{t+1} &= h_i^t + v_{i+1}^t - v_i^t \\
&\ge h_i^t + ([\alpha u_{i+2}^t + \beta] + u_{i+1}^t) - ([\alpha u_{i+1}^t + \beta] + u_i^t) \\
&\ge h_i^t + [\alpha(u_{i+2}^t - u_{i+1}^t)] + u_{i+1}^t - u_i^t \text{ (for } [a]-[b] \ge [a-b])
\end{aligned}$$

$$
\begin{aligned}
&\geq [\alpha(u_{i+2}^t - u_{i+1}^t) + u_{i+1}^t] \text{ (for } h_i^t \geq u_i^t) \\
&= [\alpha u_{i+2}^t + (1-\alpha)u_{i+1}^t] \\
&\geq \min(u_{i+1}^t, u_{i+2}^t)
\end{aligned}
$$

(ii) Just substitute Eq.(3) into (i).

(iii) $u_i^{t+1} = \min(M, h_i^{t+1}, h_i^t) \geq \min(M, u_{i+1}^t, u_{i+2}^t, h_i^t) \geq \min(u_i^t, u_{i+1}^t, u_{i+2}^t)$.

(ii)(iii) of proposition 5.2 can be understood as that there is no spontaneous jam formation when α_i and β_i are set to be the same for all i.

The Gauss function here is introduced only for changing the real number into integer, however it certainly makes some differences. It is not clear yet whether this model agrees with the realistic traffic.

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