THE MATHEMATICAL SOLUTION OF A CELLULAR AUTOMATON MODEL WHICH SIMULATES TRAFFIC FLOW WITH A SLOW-TO-START EFFECT

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Abstract

In this paper we investigate a cellular automaton model associated with traffic flow and of which the mathematical solution is unknown before. We classify all kinds of stationary states and show that every state finally evolves to a stationary state. The obtained flow-density relation shows multiple branches corresponding to the stationary states in congested phases, which are essentially due to the slow-to-start effect introduced into this model. The stability of these states is formulated by a series of lemmas, and an algorithm is given to calculate the stationary state that the current state finally evolves to. This algorithm has a computational requirement in proportion to the number of cars.

Key words: cellular automaton, traffic flow, flow-density, multiple branches, metastable states

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1. Introduction

Cellular automata[1] (CA) provide a simple, flexible way for modelling and are suitable for computer simulations. Many interesting phenomena can be observed in such simulation and then leave challenges to mathematicians. In order to study traffic flow, CA have been used extensively in recent years, and many traffic CA models have been proposed so far.[2-7]

There are basically two types of traffic CA models: Euler form and Lagrange form.[12] Models in Euler form, such as the Burger’s CA[8,10] which can be derived from Burger’s equation using an ultradiscrete method[9], focus on the number of cars at each site; while the Lagrange form models or the car-following models, such as the Nagel-Schreckenberg (NS) model[3], focus rather on the headway and velocity of each car. These two types of representations are joined...
with an Euler-Lagrange transformation,[11,12] which is a discrete version of the well-known one in hydrodynamics.

Usually the flow-density relation, or the so-called “fundamental diagram” of each traffic CA is calculated by computer simulations, and is compared with the measurements of real traffic. We show an example of real data[15] at the left of Fig.1, and point out that there is a wide scattering area near the critical point where the transition from free phase to congested phase occurs. This area suggests that there are multiple metastable states around the critical density. The flow-density relation of the model we investigate is drawn at right, which shows multiple branches really form the skeleton of such area.

Figure 1: Left: An observed flow-density relation at the Tomei expressway in Japan. Right: The flow-density relation of the model for $M = 5$. 

The model we investigate in this paper is in Lagrange form and written as:

$$x_{i}^{t+1} = x_{i}^{t} + \min(M, x_{i+1}^{t}-x_{i}^{t}-1, x_{i+1}^{t-1}-x_{i}^{t-1}-1)$$

where $M$ is a constant and $x_{i}^{t}$ denotes the position of the $i$th car at time $t$. As this is a CA model, the space and time are both discrete so $t \in \mathbb{N}$, $x_{i}^{t} \in \mathbb{Z}$. And we consider a periodic boundary condition or traffic flow on a circuit, which means that the position $x$ is identical with the position $x + L$, and that the $i$th car is identical with the $(i + N)$th car, where $L$ and $N$ represent the length of the circuit and the number of cars respectively. The constant $M$ can be understood as a legal limitation of the velocity, the term $x_{i+1}^{t}-x_{i}^{t}-1$ avoids a collision, and the term $x_{i+1}^{t-1}-x_{i}^{t-1}-1$ represents the inertia of the car or the reaction delay of the driver, which means that if $x_{i+1}^{t-1}-x_{i}^{t-1}-1 < x_{i+1}^{t}-x_{i}^{t}-1 \leq M$, then the car will maintain a low speed for an extra time step (slow-to-start effect). This kind of rule first appears in a slow-start (SIS) model proposed by Takayasu and Takayasu,[2] which is the first known deterministic two-value CA to show metastable states, its generalizations given by Nishinari both in Euler[13] and Lagrange[12] form show metastable states and multiple branches in the fundamental diagram. These multiple branches are really characteristic,
which appear in the model combined with the slow-to-start effect,[14] but are rarely observed in other models.[3,4,5] We will prove in section 2 that this effect results in many kinds of congested phases, and these phases form the multiple branches in the fundamental diagram.

Very often, a traffic CA also takes in a driver’s perspective or anticipation, which means that the \( i \)th car’s behavior depends not only on the \((i + 1)\)th car, but also the \((i + 2)\)th car and even the \((i + 3)\)th car. The slow-to-start effect and the driver’s anticipation are somehow cancelling each other, for although the driver cannot response to the fluctuation of his headway immediately, he can possibly predict the fluctuation through the headway of the car before him. So, as model (1) only takes in the slow-to-start effect, it is not amazing to find an extreme behavior of cars, which makes a jam very easy to remain and very hard to dissolve. This will be made clear in section 3 where we show some results about a jam’s “stability”. The character of such results have been observed in a model combined with the slow-to-start effect,[14] through computer simulations. There is also an algorithm to predict the final stationary state which has a computational requirement in proportion to \( N^2 \) (where \( N \) is the number of cars) given in section 3.

In section 4 we will make good use of the facts proved in section 3, and develop some tools to investigate the detailed behavior of a jam’s remaining. As a result, an algorithm with a computational requirement in proportion to \( N \) is obtained.

We conclude this paper in section 5 with a generalization of model (1) to include a driver’s perspective. Some elemental discussions about the generalized model are given.

2. Stationary states and the flow-density relation

**Notation 2.1.** By \( C_i \) we mean the \( i \)th car and \( C_i^t \) the \( i \)th car at time \( t \). \( x_{i+1}^t - x_i^t \) is called the velocity of \( C_i^t \) and denoted by \( v_i^t \). \( x_{i+1}^t - x_i^t - 1 \) is called the headway of \( C_i^t \) and denoted by \( h_i^t \). Eq.(1) is then rewritten as:

\[
v_i^t = \min(M, h_i^t, h_i^{t-1})
\]

with \( h_i^{t+1} = h_i^t + v_{i+1}^t - v_i^t \) by definition. Immediately we get \( h_i^t \geq v_i^t \), \( h_i^{t-1} \geq v_i^t \).

There are two apparent types of stationary states which satisfy this equation, namely the “free state” that \( v_i^t = M, h_i^t \geq M \) for all \( i \); and the “\( \tau \)-uniform state” that \( v_i^t = h_i^t = \tau < M \) for all \( i \). Note that if we have a uniform state at time \( t \), which means \( v_i^t = h_i^t \) for all \( i \), then \( h_i^{t-1} \geq v_i^t = h_i^t \) for all \( i \), however \( \sum_{i=1}^N h_i^t = L - N \) is a constant, so we conclude that all the inequalities should be equalities and the state at time \( t - 1 \) must also be a uniform state. This shows that the uniform state is very unstable.

**Lemma 2.2.** The following inequalities hold:

(i) \( h_i^{t+1} \geq v_{i+1}^t \)
(ii) \( h_i^{t+1} \geq \min(M, h_{i+1}^t, h_i^{t-1}) \)

(iii) \( v_i^{t+1} \geq \min(v_{i+1}^t, v_i^t) \)

**Proof.**

(i) 
\[
h_i^{t+1} = h_i^t + v_i^{t+1} - v_i^t \quad \text{(by definition)}
\geq v_i^{t+1} \quad \text{(for } h_i^t \geq v_i^t)\]

(ii) by (i) we have \( h_i^{t+1} \geq v_i^{t+1} = \min(M, h_{i+1}^t, h_i^{t-1}) \).

(iii)
\[
v_i^{t+1} = \min(M, h_{i+1}^t, h_i^t)
\geq \min(M, v_{i+1}^t, h_i^t) \quad \text{(for } h_i^{t+1} \geq v_i^{t+1} \text{ by (i)})
\geq \min(v_{i+1}^t, \min(M, h_i^{t+1}, h_i^{t-1}))
= \min(v_{i+1}^t, v_i^t) \quad \text{(by Eq.(2))}\]

**Corollary 2.3.** \( \min\{v_i^t | i \in \mathbb{Z}\} \) (the minimal velocity of cars at time \( t \)) is a nondecreasing function of \( t \).

**Proof.** Use (iii) of lemma 2.2.

Briefly speaking, this corollary shows that there is no spontaneous jam formation in model (1).

**Notation 2.4.** We use \( \tau_t \) to denote this \( \min\{v_i^t | i \in \mathbb{Z}\} \).

Obviously, the slow-to-start effect appears in an accelerating process and result in a nontrivial low speed. The definition below focuses on this phenomenon and clarifies the main object we investigate in this paper.

**Definition 2.5.** If one of the following conditions is satisfied, we call \( C_i^t \) an \( \alpha \)-front for \( 0 \leq \alpha < M \).

(i) \( v_i^t = h_i^t = \alpha, v_{i+1}^t > \alpha \)

(ii) \( v_i^t = \alpha, h_i^t > \alpha \)

Note that the two conditions are incompatible. We say the front is **type A** when (i) is satisfied and **type B** when (ii) is satisfied.

**Lemma 2.6.** If \( C_i^t \) is an \( \alpha \)-front, then \( C_i^{t-1} \) or \( C_i^{t-1} \) is a \( \beta \)-front with \( \beta \leq \alpha \). More precisely, when \( C_i^t \) is a type A \( \alpha \)-front, \( C_i^{t+1} \) must be a type B \( \gamma \)-front with \( \gamma \leq \alpha \). In the case \( C_i^t \) is type B, we have \( C_i^{t-1} \) as a type A \( \alpha \)-front or a type B \( \delta \)-front with \( \delta < \alpha \).

**Proof.**

(i) When \( C_i^t \) is type A:

Let \( \gamma = v_{i+1}^{t-1} \), and we have \( \alpha = h_i^t \geq v_{i+1}^{t-1} = \gamma, h_i^{t-1} \geq v_{i+1}^t > \alpha \geq \gamma \), so \( C_i^{t+1} \) is a type B \( \gamma \)-front.
(ii) When $C_t^i$ is type B:

We have $v_t^i = \alpha < M, h_t^i > v_t^i$ so $v_t^i = \alpha$ must be equal to $h_t^{i-1}$ since $v_t^i = \min(M, h_t^i, h_{t-1}^i)$. Then (a) $v_t^{i-1} = \alpha$ or (b) $v_t^{i-1} < \alpha$. If $v_t^{i-1} = \alpha$, then $\alpha < h_t^i = h_t^{i-1} - v_t^{i-1} + v_{t+1}^{i-1} = v_t^{i+1}$ and we can say $C_t^{i-1}$ is a type A $\alpha$-front. If $\delta = v_t^{i-1} < \alpha$, the condition for $C_t^{i-1}$ to be a type B $\delta$-front is then satisfied.

**Definition 2.7.** Let $F_t$ be the set consists of all fronts at time $t$. Lemma 2.6 suggests that we can define a front map $\phi_t : F_t \rightarrow F_{t-1}$ like this:

For $C_t^i \in F_t$, let $\phi_t(C_t^i) = \begin{cases} C_t^{i-1} & \text{if } C_t^i \text{ is type A} \\ C_t^{i-1} & \text{if } C_t^i \text{ is type B} \end{cases}$

We also define $G_t$ to be the set consists of all $\tau_t$-fronts at time $t$.

An example is shown in Fig.2 with $L = 40, N = 9, M = 4$, from time point 0 to 10. We can see the concept “front” somehow corresponds to “the front of a jam”, and the front map somehow joins the fronts of “the same jam”.

![Figure 2: An example. The 9 cars are denoted by numbers from 1 to 9, or by alphabet A (resp. B) when it is a type A (resp. type B) front.](image)

**Lemma 2.8.** $\phi_t|_{G_t} : G_t \rightarrow F_{t-1}$ (the restriction of $\phi_t$ to $G_t$) is an injection.

**Proof.** If this is not true, we can assume that there exist $C_t^i$ and $C_{t+1}^i$ in $G_t$ such that $\phi_t(C_t^i) = \phi_t(C_{t+1}^i)$. That means $C_t^i$ is a type A $\tau_t$-front and $C_{t+1}^i$ a type B $\tau_t$-front. $C_t^i$ to be a type A $\tau_t$-front implies $v_{t+1}^i > \tau_t$ while $C_{t+1}^i$ to be a type B $\tau_t$-front implies $v_{t+1}^i = \tau_t$. That is a contradiction.

Note that if $\tau_t = \tau_{t-1}$, then by lemma 2.6 the image of $\phi_t|_{G_t}$ lies in $G_{t-1}$. $\tau_t$ is a nondecreasing function of $t$ by Corollary 2.3, and it is an integer less than $M$, so...
it becomes a constant when \( t \gg 0 \). We write the constant \( \tau_\infty \). If \( \tau_\infty = M \), the traffic is free. If \( \tau_\infty < M \), \( G_t \) with \( t \gg 0 \) is empty precisely when \( h^t_i = v^t_i = \tau_\infty \) for all \( i \). This uniform state is trivial so we assume that \( G_t \neq \emptyset \). Then lemma 2.8 implies that \#\( G_t \) (the cardinality of \( G_t \)) is non-increasing when \( t \gg 0 \). That means we can assume \#\( G_t \) to be a positive constant when \( t \gg 0 \).

**Definition 2.9.** Let \( C^t_i \) be a \( \tau \)-front. The **preceding \( \tau \)-front** of \( C^t_i \) is the \( \tau \)-front \( C^t_j \) with \( j > i \) such that no other \( \tau \)-front exists between \( C^t_i \) and \( C^t_j \). We say \( C^t_i \) connects properly to a \( \tau \)-front \( C^t_j \) if for all \( i < k < j \), the following conditions are satisfied:

1. If \( C^t_i \) is type B, then \( h^t_i \leq M \) and \( C^t_{i+1} \) is not a type B \( \tau \)-front.
2. If \( v^t_{k+1} = \tau \), then the following (a) or (b) holds:
   - (a) \( \tau < h^{t+1}_k = h^t_k \tau_M = v^t_k \leq M \) and \( h^t_{k+1} = \tau \).
   - (b) \( h^t_k = v^t_k \geq \tau \).
3. If \( v^t_{k+1} \neq \tau \), then \( h^t_k + \tau - v^t_k = M \).
4. If \( k > i + 1 \) or \( C^t_i \) is type B, then \( v^t_k \) should be \( M \) unless the (2b) case.

**Lemma 2.10.** If a \( \tau \)-front \( C^t_i \) connects properly to a \( \tau \)-front \( C^t_j \), then \( v^t_k \geq \tau \) for all \( i < k < j \), and \( C^t_j \) is the preceding front of \( C^t_i \).

**Proof.** When \( k = i + 1 \) and \( C^t_i \) is type A, then \( v^t_k > \tau \) by definition. In the case \( k > i + 1 \) or \( C^t_i \) is type B, condition (4) and (2b) implies \( v^t_k \geq \tau \). Now we assume \( v^t_k = \tau \). Condition (4) says that this could happen only if \( v^t_k = h^t_k = \tau = v^t_{k+1} \). So \( C^t_k \) is not a \( \tau \)-front.

Roughly speaking, a proper connection structure looks like several successive cars with headway \( \tau \) and velocity \( \tau \) preceding several successive cars with headway \( 2M - \tau \) and velocity \( M \). The next lemma shows that this structure somehow repeats itself after two time steps.

**Lemma 2.11.** If a \( \tau \)-front \( C^t_i \) connects properly to its preceding \( \tau \)-front \( C^t_j \), then for all \( i \leq k < j \), we have \( x^{t+2}_k = x^{t+1}_{k+1} + \tau - 1, v^{t+2}_k = v^{t+1}_{k+1} \). Moreover, if \( C^t_{p+1} \) and \( C^t_{q+1} \) are \( \tau \)-fronts satisfy \( \phi_{t+1}(C^t_{p+1}) = C^t_{i+1}, \phi_{t+1}(C^t_{q+1}) = C^t_{j+1} \), then \( C^t_{p+1} \) connects properly to \( C^t_{q+1} \).

**Proof.** First we use (iii) of lemma 2.2 to get \( v^{t+1}_{k+1} \geq \min(v^{t+1}_{k+1}, v^{t+1}_k) \geq \tau \). We prove the first part of this lemma by dividing it into the following cases. In each case we basically calculate \( h^{t+1}_k \) by \( h^{t+1}_k = h^t_k + v^t_k - v^{t+1}_k \) and \( v^{t+1}_k \) by Eq.(2).

\( x^{t+2}_k = x^{t+1}_{k+1} + \tau - 1 \) holds precisely when \( v^t_k + v^{t+1}_k = h^t_k + \tau \). Then we estimate \( h^{t+2}_k \) by \( h^{t+2}_k = h^{t+1}_k - v^{t+1}_k + v^{t+2}_k \geq h^{t+1}_k - v^{t+1}_k + \tau \), and get \( v^{t+2}_k \) by Eq.(2).

(i) \( k = i \) and \( C^t_i \) is type A:

Note the definition of a type A front and we have \( h^{t+1}_i = v^{t+1}_{i+1}, v^{t+1}_i = \tau \) and \( v^t_i + v^{t+1}_i = h^t_i + \tau \) can be checked. \( h^{t+2}_i \geq h^{t+1}_i \) so \( v^{t+2}_i = v^{t+1}_i + \tau \).
(ii) \( k = i \) and \( C_i^t \) is type B:

The definition of a type B front implies \( h_t^{i+1} = h_t^i + v_t^i + 1 - \tau \geq h_t^i \), condition (1) says \( M \geq h_t^i \), so \( v_t^i + 1 = h_t^i \) and \( v_t^i + v_t^{i+1} = h_t^i + \tau \) can be checked. If \( i + 1 = j \), then \( C_j^t \) must be type A, so we have \( v_t^i + 1 = v_t^{i+1} = \tau \), then \( h_t^{i+1} = h_t^i, h_t^{i+2} = v_t^{i+2} = \tau = v_t^i + 1 \). If \( i + 1 < j \), by condition (4), \( v_t^{i+1} = M \) or \( C_{i+1}^t \) satisfies (2b). In the former case, \( h_t^{i+1} \geq M, h_t^{i+2} \geq M \) so \( v_t^{i+2} = M = v_t^{i+1} \). In the latter case, we have \( v_t^{i+1} = \tau \) so \( h_t^{i+2} = v_t^{i+1} \) and \( v_t^{i+2} = h_t^{i+2} = v_t^i + 1 \).

(iii) \( k \geq i + 1, v_{k+1}^t \neq \tau \):

If condition (2a) is satisfied, we have \( h_t^{i+1} \leq M \) and \( h_t^{i+1} \leq h_t^k \) since \( v_t^k \geq \tau \).

So \( h_t^{i+1} = h_t^k + \tau - v_t^k \). From \( h_t^{k+1} = \tau \) and \( v_t^{k+2} \geq \tau \) we calculate \( v_t^{k+1} + 1 = \tau \) so \( h_t^{k+2} = \tau \) and \( v_t^{k+2} = \tau \). If condition (2b) is satisfied, we have \( h_t^{k+1} = \tau \) and \( v_t^{k+1} = \tau \); \( h_t^{k+2} \geq \tau \) so \( v_t^{k+2} = \tau = v_t^t \).

The condition (4) says that \( v_t^{k+1} = M \) or \( C_{k+1}^t \) satisfies (2b). In the former case we have \( h_t^{k+1} = 2M - \tau \) by condition (3), then \( v_t^{k+1} = M, h_t^{k+2} \geq M \) so \( v_t^{k+2} = M = v_t^{k+1} \). In the latter case, \( h_t^{k+1} = M - \tau + v_t^{k+1} = M \) and \( v_t^{k+1} = M \). Since \( C_{k+1}^t \) satisfies (2b), we have just showed in (iii) that \( v_t^{k+1} = \tau \), which implies \( h_t^{k+2} = v_t^{k+1} \) and so \( v_t^{k+2} = v_t^{k+1} \).

To prove the second part of this lemma, we simply check the four conditions for all \( p < k < q \). (1) If \( C_t^{p-1} \) is type B, that means \( p = i \) and \( C_t^i \) is type A by lemma 2.6. So \( h_t^{i+1} = \tau \). If \( v_t^{i+1} = \tau \), then \( h_t^{i+1} \) must be \( \tau \) since \( h_t^{i+1} \geq \tau \) and \( \tau = v_t^{i+1} = \min(M, h_t^{i+1}, h_t^{i+1}) \). That means \( C_t^{p-1} \) cannot be a type B \( \tau \)-front. (2)(3)(4) As \( p < q \) implies \( i \leq k < j \), we summarize the preceding calculation results here: If (ii) and \( i + 1 = j \), we have \( v_t^{k+1} = h_t^k \) and \( v_t^{k+1} = \tau \); If (ii) and \( i + 1 < j \), we have \( h_t^{i+1} = v_t^{i+1} + \tau = v_t^{k+1} \), with \( v_t^{k+1} = M \) or \( v_t^{k+1} = h_t^k + 1 \) and \( v_t^{k+1} = \tau \). (iii) we have \( v_t^{i+1} = h_t^k \) and \( v_t^{i+1} = \tau \). (iv) we have \( v_t^{i+1} = M, h_t^{i+1} = v_t^{i+1} + \tau = v_t^{k+1} \), with \( v_t^{k+1} = M \) or \( v_t^{k+1} = h_t^k + 1 \). (Note that \( C_t^i \) must be type B when \( k = i \), and we can assume \( h_t^k + 1 = \tau \) if \( v_t^{k+1} = \) \( \tau \), for otherwise \( C_t^k \) is a type B \( \tau \)-front so we have \( k + 1 = j \) and \( k = q = j - 1 \).

In any case, (2)(3)(4) hold.

**Definition 2.12.** A \( \tau \)-congested state is a state that contains at least one \( \tau \)-front and all its \( \tau \)-fronts connect properly to their preceding \( \tau \)-fronts.

**Corollary 2.13.** If we have a \( \tau \)-congested state at time \( t \), then the state at time \( t + 1 \) is also a \( \tau \)-congested state, and \( x_t^{i+2} = x_t^{i+1} + \tau - 1, v_t^{i+2} = v_t^{i+1} \) for all \( i \).

**Proof.** Take an arbitrary \( \tau \)-front \( C_t^i \). By lemma 2.11, what to prove is that we can find a \( \tau \)-front \( C_t^{p-1} \) satisfies \( \phi_t^{i+1}(C_t^{p-1}) = C_t^i \). This can always be done when \( C_t^i \) is type A since then \( C_t^{p-1} \) is automatically a type B \( \tau \)-front. Now assume \( C_t^i \) to be type B. \( C_t^i \) is properly connected, so we can apply condition (2) to \( C_{i-1}^t \) and get \( h_t^{i-1} = v_t^{i-1} + \tau \), that means \( h_t^{i-1} = v_t^{i+1} = \tau \). On the other hand, lemma 2.10 says \( v_t^{i+1} \geq \tau \), so \( h_t^{i+1} = h_t^i - \tau + v_t^{i+1} \geq h_t^i > \tau \), and hence \( v_t^{i+1} > \tau \). Now we can say that \( C_t^{i+1} \) is a type A \( \tau \)-front.
Using the density $\rho = N/L$, we can represent the flow $Q$ of a $\tau$-congested state by $\rho$ as:
\[
Q = \frac{1}{\tau} \sum_{i=1}^{N} (x_{i}^{t+2} - x_{i}^{t}) = \frac{1}{\tau} \sum_{i=1}^{N} (x_{i}^{t+1} - x_{i}^{t} + \tau - 1) = \frac{N(\tau-1)+L}{2\tau} = \frac{\tau-1}{2} \rho + \frac{1}{2}.
\]
Since the headway of a car in a $\tau$-congested state is always between $\tau$ and $2M - \tau$, we have
\[
\frac{1}{2M - \tau + 1} < \rho < \frac{1}{\tau + 1}.
\]

**Theorem 2.14.** Every state finally evolves to one of the followings:

(i) a free state.

(ii) a $\tau$-uniform state.

(iii) a $\tau$-congested state.

**Proof.** As we have discussed above, if not the (i) or (ii) case, we can assume that $\tau_{t} = \tau_{\infty} = \tau$ and $\#G_{t}$ is a positive constant at time $t \gg 0$. Now we prove it to be the (iii) case. Since $\#G_{t}$ is a constant, if we have a type B $\tau$-front at time $t$ there must be a corresponding type A $\tau$-front at time $t + 1$ (and vice versa). Then by lemma 2.11, once at time $t$ a $\tau$-front $C_{i}^{t}$ connects properly to its preceding $\tau$-front, there will always be a $\tau$-front corresponds to $C_{i}^{t}$ which also connects properly to its preceding $\tau$-front after $t$. This verifies that we can only consider a type A $\tau$-front $C_{i}^{t}$ and its correspondences at time $t + 2, t + 4, \text{etc.}$, to see if it connects properly to its preceding $\tau$-front at last. We enumerate some cases at the beginning in which the type A $\tau$-front $C_{i}^{t}$ connects to its preceding just properly:

- $h_{i+1}^{t} \leq M + v_{i+1}^{t} - \tau, v_{i+2}^{t} = \tau$.
  In this case, if $h_{i+1}^{t} = v_{i+1}^{t}$, then the condition (2b) in definition 2.9 is satisfied for $k = i + 1$; otherwise if $h_{i+1}^{t} > v_{i+1}^{t}$, we have $h_{i+2}^{t} = \tau$ since the type B $\tau$-front $C_{i+2}^{t}$ will not have a correspondence at time $t + 1$ if $h_{i+2}^{t} > \tau$, so the condition (2a) in definition 2.9 is satisfied for $k = i + 1$.

- $h_{i+1}^{t} = M + v_{i+1}^{t} - \tau, v_{i+2}^{t} = h_{i+2}^{t}, v_{i+3}^{t} = \tau$.
  In this case condition (3) in definition 2.9 is held for $k = i + 1$.

- $h_{i+1}^{t} = M + v_{i+1}^{t} - \tau, v_{i+2}^{t} = M, h_{i+2}^{t} \leq 2M - \tau, v_{i+3}^{t} = \tau$.
  In this case if $h_{i+2}^{t} > v_{i+2}^{t}$, then $h_{i+3}^{t}$ must be $\tau$, for otherwise the type B $\tau$-front $C_{i+3}^{t}$ will not have a correspondence at time $t + 1$. So condition (2a) is satisfied for $k = i + 2$.

- $h_{i+1}^{t+1} = M + v_{i+1}^{t+1} - \tau; j \geq i + 2, v_{k}^{t} = M, h_{k}^{t} = 2M - \tau$ for all $i + 2 \leq k \leq j$; $v_{j+1}^{t} = h_{j+1}^{t}, v_{j+2}^{t} = \tau$.

- $h_{i+1}^{t+1} = M + v_{i+1}^{t+1} - \tau; j \geq i + 2, v_{k}^{t} = M, h_{k}^{t} = 2M - \tau$ for all $i + 2 \leq k \leq j$; $v_{j+1}^{t} = M, h_{j+1}^{t} \leq 2M - \tau, v_{j+2}^{t} = \tau$.

The last two are similar to the above. Now before considering $C_{i}^{t}$ in the following cases, let $f$ denote $v_{i+1}^{t}$, and confirm that $C_{i}^{t}$ is correspondent to $C_{i-1}^{t+2}, C_{i-2}^{t+4}, C_{i-3}^{t+6}, \text{ etc.}$ with $v_{i-1}^{t+2} = v_{i-2}^{t+4} = v_{i-3}^{t+6} = \ldots = f$. 

8
(i) \( h_{i+1}^t > M + f - \tau \):
We have \( h_{i+1}^{t+1} > h_{i+1}^t + \tau - v_{i+1}^t > M \), so \( v_{i+1}^{t+1} = M \), and we calculate \( h_{i+1}^{t+2} = M + f - \tau \). That means the correspondence \( C_{i+1}^{t+2} \) has \( h_{i+1}^{t+2} = M + f - \tau \) and hence the (iii) case.

(ii) \( h_{i+1}^t < M + f - \tau \):
If \( v_{i+2}^t = \tau \), \( C_i^t \) connects to its preceding properly. If \( v_{i+2}^t > \tau \), we have \( h_{i+1}^{t+1} > h_{i+1}^t + \tau - f \), and \( h_{i+1}^t > h_{i+1}^t + \tau - \tau \) since \( v_{i+1}^t > \tau \), and \( M > h_{i+1}^t + \tau - \tau \) since \( h_{i+1}^t < M + f - \tau \). So \( v_{i+1}^{t+1} > h_{i+1}^t + \tau - f \) and we calculate \( h_{i+2}^{t+2} = h_{i+1}^t \). That means the correspondences of \( h_{i+1}^t \) at time \( t+2, t+4, t+6, \ldots \) (i.e. \( h_{i+2}^{t+2}, h_{i+4}^{t+4}, h_{i+6}^{t+6}, \ldots \)) strictly increase until \( M + f - \tau \) or until the correspondent front connects to its preceding properly.

(iii) \( h_{i+1}^t = M + f - \tau \):
First check that this condition will be inherited by \( v_{i+1}^{t+2} = v_{i+1}^{t+1} + \tau - f \), and \( h_{i+1}^{t+2} = h_{i+1}^t + \tau - f \) since \( v_{i+1}^t > \tau \), and \( M > h_{i+1}^{t+2} \) since \( h_{i+1}^t < M + f - \tau \). So \( v_{i+1}^{t+1} > h_{i+1}^t + \tau - f \) and we calculate \( h_{i+2}^{t+2} = h_{i+1}^t \). That means the correspondences of \( h_{i+1}^t \) at time \( t+2, t+4, t+6, \ldots \) (i.e. \( h_{i+2}^{t+2}, h_{i+4}^{t+4}, h_{i+6}^{t+6}, \ldots \)) strictly increase until \( M + f - \tau \) or until the correspondent front connects to its preceding properly.

Anyway, we will finally find a correspondence \( C_{i-n}^{t+2n} \) such that \( h_{i+1}^{t+2n} = 2M - \tau \), \( v_{i+1}^{t+2n} = M \) if \( C_{i-m}^{t+2m} \) did not connect to its preceding properly for all \( 0 \leq m \leq n \). Next we consider the general case:
\( v_k^t = M \), \( h_k^t = 2M - \tau \) for all \( i + 2 \leq k \leq j \) with a \( j \geq i + 2 \).
First check that this condition will be inherited by \( C_{i+1}^{t+2} \) as \( v_{i+1}^{t+2} = M \), \( h_{i+1}^{t+2} = 2M - \tau \) for all \( i + 2 \leq k \leq j \). Similarly we can assume \( v_{j+1}^t = M \) and have \( v_{j+1}^{t+1} = \min(M, h_{j+1}^t) \). Consider the following cases and calculate just as above:

(a) \( h_{j+1}^t > 2M - \tau \):
We have \( h_{j+1}^{t+2} = 2M - \tau \), \( v_{j+1}^{t+2} = M \).

(b) \( M \leq h_{j+1}^t < 2M - \tau \):
If \( v_{j+2}^t = \tau \), \( C_i^t \) connects properly. If \( v_{j+2}^t > \tau \), we have \( h_{j+2}^{t+2} > h_{j+1}^t \).

(c) \( h_{j+1}^t < M \):
If \( v_{j+3}^t = \tau \), \( C_i^t \) connects properly. If \( v_{j+3}^t > \tau \), we have \( h_{j+3}^{t+2} > h_{j+1}^t \).
So if \( C_i^t \) has not connected properly to its preceding yet, we can then consider a correspondence \( C_{i-n}^{t+2n} \) such that \( v_k^{t+2n} = M, h_k^{t+2n} = 2M - \tau \) for all \( i + 2 \leq k \leq j + 1 \).

Since the number of cars is limited, we conclude that \( C_i^t \) will finally connects properly to its preceding \( \tau \)-front.

Using theorem 2.14, we can now calculate the flow-density relation at time \( t > 0 \). Obviously, a free state has a \( Q = M\rho \) for \( \rho \leq \frac{1}{M-1} \) and a \( \tau \)-uniform state has a \( Q = \tau \rho \) for \( \rho = \frac{1}{\tau + 1} \). And we have calculated the flow-density relation of a \( \tau \)-congested state as \( Q = \frac{\tau - 1}{2} \rho + \frac{1}{2} \) for \( \frac{1}{2M-\tau+1} < \rho < \frac{1}{\tau + 1} \). This is the result we have shown in Fig.1.

3. Stabilities and an \( \text{O}(N^2) \) algorithm

The next two lemmas show a remarkable character of model (1).

**Lemma 3.1.** Assume \( v_i^{t-1} \geq \alpha \). Then \( h_i^t \leq 2M - \alpha \) implies \( h_{i+1}^{t+1} \leq 2M - \alpha \).

**Proof.** Consider the three different cases of \( v_i^t \). (i) \( v_i^t = h_i^t \). Then \( h_{i+1}^t = h_i^t + v_{i+1}^t = v_i^t + v_{i+1}^t \leq M \). (ii) \( v_i^t = M \). Then \( h_{i+1}^t = h_i^t + v_{i+1}^t - M \leq h_i^t \leq 2M - \alpha \).

(iii) \( v_i^t = h_{i-1}^t \). We have \( h_i^t = h_{i-1}^t + v_{i+1}^t - v_i^t \leq h_{i-1}^t + M - \alpha = v_i^t + M - \alpha \) by assumption. So \( h_{i+1}^t \leq h_i^t + M - v_i^t \leq 2M - \alpha \).

**Lemma 3.2.** Suppose we have a type A \( \alpha \)-front \( C_i^t \) and assume that \( v_{i-1}^{t-1} \geq \alpha, v_{i+1}^{t+1} \geq \alpha \). Then the necessary and sufficient condition for \( C_i^{t+2} \) to be a type A \( \alpha \)-front is \( \alpha \leq h_i^{t-1} \leq 2M - \alpha \).

**Proof.** Necessity: If \( \alpha > h_i^{t-1} \), we have \( v_{i-1}^{t+1} < \alpha \) so \( h_{i-1}^{t+2} > h_{i-1}^{t+1} \) which means \( h_{i-1}^{t+2} < v_{i-1}^{t+2} \) and hence \( C_i^{t+2} \) is not a type A front. If \( 2M - \alpha < h_i^{t-1} \), we have \( h_i^{t+2} > \alpha \) since \( v_i^{t-1} \leq M, v_i^{t+1} \leq M \). So \( C_i^{t+2} \) is not a type A front.

Sufficiency: First note that \( v_{i+1}^{t+1} \geq \alpha \) implies \( h_{i+1}^{t+2} > h_{i+1}^t \) and since \( h_{i+1}^t = v_i^{t+1} = \alpha \) we get \( v_{i+1}^{t+2} > \alpha \). Next we consider the following two situations and show in each case \( h_i^{t-1} = v_i^{t-1} = \alpha \).

(i) \( \alpha \leq h_i^{t-1} \leq M \).
This case, we have \( v_i^{t-1} \geq \alpha \) since both \( h_i^{t-1} \geq v_i^{t-1} \geq \alpha \) and \( h_i^{t-1} \geq \alpha \).
So we calculate \( h_i^{t-1} \leq h_i^{t-1} \leq M \) which implies \( v_i^{t+1} = h_i^{t+1} = v_i^{t+2} = \alpha \).

(ii) \( M \leq h_i^{t-1} \leq 2M - \alpha \).
If \( v_i^{t-1} = M \) we have \( h_i^{t+1} \leq M \). Or if \( v_i^{t-1} = h_i^{t-1} \), then \( h_i^{t-1} = h_i^{t-1} + v_i^{t-1} - v_i^{t-1} \leq h_i^{t-1} + M - \alpha = v_i^{t-1} + M - \alpha \), so again \( h_i^{t+1} = h_i^{t-1} + M - \alpha = v_i^{t-1} + M - \alpha \).
Since \( h_i^{t-1} \geq M \), we have \( v_i^{t+1} = h_i^{t+1} = h_i^{t+2} = v_i^{t+2} = \alpha \).

Combining lemma 3.1 with lemma 3.2 we get the next lemma.
Lemma 3.3. Suppose $C_i^{t+k}$ to be a type A $\alpha$-front with $k \geq 1$. Assume that $v_{i+1}^{t+k+1} \geq \alpha$ and $v_{i-1}^{t+s} \geq \alpha$ for all $0 \leq s \leq k$. If there exists $1 \leq r \leq k$ such that $h_{i-1}^{x-1} \leq 2M - \alpha$, then $C_i^{t+k+2}$ is a type A $\alpha$-front.

The proof is obvious.

Lemma 3.4. Suppose $C_i^{t+k}$ to be a type A $\alpha$-front with $k \geq 0$. Assume that $v_{i+1}^{t+k+1} \geq \alpha$ and $v_{i-1}^{t+s} \geq \alpha$ for all $0 \leq s \leq k$. Then the necessary and sufficient condition for $C_i^{t+k+2}$ to be a type A $\alpha$-front is that $x_i^{t+k} - x_{i-1}^{t-1} - 1 \leq v_{i-1}^{t} + (k + 1)M - \alpha$.

Proof. Necessity: It is easy to check that if $C_i^{t+k+2}$ is a type A $\alpha$-front then $x_i^{t+k} - x_{i-1}^{t-1} - 1 \leq v_{i-1}^{t} + (k + 1)M - \alpha$. On the other hand we have $x_i^{t+k+2} = x_i^{t+k} - M \geq h_{i-1}^{x-1} - M$. Also $v_{i-1}^{t} \geq v_{i-1}^{t-1} + (k + 1)M - \alpha$ for all $0 \leq s \leq k$.

Sufficiency: In the case $k = 0$, the assumption is $v_{i-1}^{t+1} \geq \alpha$, $h_{i-1}^{x-1} \geq v_{i-1}^{t}$ and $M - \alpha, v_{i-1}^{t} \geq \alpha$. First note that $v_{i-1}^{t+1} \geq \alpha$ implies $v_{i-1}^{t+k+2} \geq h_{i-1}^{x-1}$. If $h_{i-1}^{x-1} \leq M$, we have $h_{i-1}^{x-1} = h_{i-1}^{x-1}$. If $h_{i-1}^{x-1} > M$, we have $h_{i-1}^{x-1} = h_{i-1}^{x-1}$. Hence $v_{i-1}^{t+1} \geq M - \alpha$ and then again $v_{i-1}^{t+1} = h_{i-1}^{x-1}$. Anyway, $v_{i-1}^{t+1} = h_{i-1}^{x-1}$. If $k \geq 1$, assume $C_i^{t+k+2}$ is not a type A $\alpha$-front. Then by lemma 3.3 we have $h_{i-1}^{x-1} > 2M - \alpha$ for all $1 \leq r \leq k$. In particular, we have $h_{i-1}^{x-1} = h_{i-1}^{x-1} + (k + 1)M - \alpha$ for all $1 \leq r \leq k$. That means $x_i^{t+k} = 1 - x_{i-1}^{t+1} + h_{i-1}^{x-1} > (x_{i-1}^{t} + v_{i-1}^{t} + (k - 1)M) + (2M - \alpha)$ which leads to a contradiction.

Remark 3.5. In fact lemma 3.4 is true even when $k = -1$. For this case, the expression $x_i^{t+k} - x_{i-1}^{t-1} - 1 \leq v_{i-1}^{t} + (k + 1)M - \alpha$ becomes $h_{i-1}^{x-1} \geq v_{i-1}^{t}$, on the other hand we have $h_{i-1}^{x-1} \geq v_{i-1}^{t}$ and $h_{i-1}^{x-1} = v_{i-1}^{t}$, so it is equivalent to $h_{i-1}^{x-1} \geq v_{i-1}^{t}$, which is the necessary and sufficient condition for $C_i^{t+1}$ to be a type A $\alpha$-front.

Furthermore, if $C_i^{t+k+2}$ is not a type A $\alpha$-front, the proof of lemma 3.4 actually shows that $v_{i-1}^{t+r} = M$ for all $1 \leq r \leq k + 1$. So this case we have $x_i^{t+k+2} = x_i^{t+k} = v_{i-1}^{t} + (k + 1)M$ for $k \geq -1$, and $v_{i-1}^{t+k+2} = min(M, h_{i-1}^{x-1}) + 2M$ for $k \geq 0$.

Definition 3.6. We say a type A $\alpha$-front $C_i^{t}$ remains to $C_{i-k}$ if for all $0 \leq j \leq k$, $C_{i-j}^{t+2j}$ is a type A $\alpha$-front.

If we somehow know the information about a type A $\alpha$-front $C_i^{t+k}$ at a future time $t + k$, then lemma 3.4 provides a way for us to predict whether the front remains or not through the information at the current time $t$. If the front remains to $C_{i-1}$, we get the information about $C_i^{t+k+2}$; Even not, we also have $h_{i-1}^{x-1}$ and $v_{i-1}^{t+k+2}$ by remark 3.5. This will be actively used in section 4 to investigate the detailed behavior of a front. Here we only pick up an outstanding specialization.
Corollary 3.7. Suppose $C_i^t$ to be a type $A$ $\tau_t$-front. Then $C_i^t$ remains to $C_{i-k}$ precisely when $x_i^t - x_i^{t-j} \leq j(2M - \tau_t + 1) + v_i^{t-j} - M$ for all $0 \leq j \leq k$.

Proof. Since we are thinking about a $\tau_t$-front, lemma 3.4 can be freely used without care about the estimate of velocities. Note that if $C_i^t$ remains to $C_{i-j}$, then we have $x_i^{t+j} = x_i^t + j(\tau_t - 1)$, and $C_k^t$ remains to $C_{1-j}$ precisely when $x_i^{t+j} - x_i^{1-j} - 1 \leq v_i^{1-j} + (2j + 1)M - \tau_t$ by lemma 3.4. That is the expression $x_i^t + j(\tau_t - 1) - x_i^{t-1} - 1 \leq v_i^{t-1} + (2j + 1)M - \tau_t$ or $x_i^t - x_i^{t-1} - 1 \leq (j + 1)(2M - \tau_t + 1) + v_i^{t-1} - M$.

Roughly speaking, this corollary means that we cannot untie an $l$-car-long $\tau$-jam unless there is an $(l(2M - \tau + 1)$ gap. In particular, the smaller $\tau$ is, the more difficult we untie the jam. Also, even one car with headway $\tau$ can propagate a jam in not so strict conditions. This can somehow be broken if we introduce a driver’s perspective, but random generated, rather uniformly distributed cars will almost always cluster behind the car with the narrowest headway.[14]

Theorem 3.8. If $\tau_{t+2N} = \tau_t$, then $\tau_\infty = \tau_t$.

Proof. If it is a free state or a uniform state at time $t + 2N$, the statement is certainly true. So we assume $G_{t+2N} \neq \emptyset$. Let $\tau = \tau_t$, take a $\tau$-front $C_{k}^{t+2N}$. If $C_{k}^{t+2N}$ is type $B$, since $\tau_{t+2N} = \tau_{t+2N-1}$, $\phi_{t+2N}(C_{k}^{t+2N})$ is a type $A$ $\tau$-front by lemma 2.6. So it does not matter for us to change the assumption to: $\tau_{t+2N-2} = \tau_t$ and there is a type $A$ $\tau$-front in $G_{t+2N-2}$. Under this assumption, take a type $A$ $\tau$-front $C_{k}^{t+2N}$, let $C_{i}^t = \phi_{t+1} \circ \cdots \circ \phi_{t+2N-3} \circ \phi_{t+2N-2}(C_{k}^{t+2N-2})$. By lemma 2.6, $C_{i}^t$ must be a type $A$ $\tau$-front, and $C_{i}^t$ remains to $C_{k}^t$.

Then by corollary 3.7, we have $x_i^t - x_i^{t-j} \leq j(2M - \tau + 1) + v_i^{t-j} - M$ for all $0 \leq j \leq N - 1$. In particular, $x_i^t - x_i^{t-N+1} \leq (N - 1)(2M - \tau + 1)$. By the periodic boundary condition, $C_{i-N}^t$ is a type $A$ $\tau$-front identical with $C_{i}^t$, so we have $x_i^{t-N+1} = x_i^{t-N} + \tau = x_i^{t-N} + L + \tau$. This implies $L \leq N(2M - \tau + 1)$. Now take an arbitrary integer $pN + q \geq 0$ with $0 \leq q \leq N - 1$, use the periodic boundary condition we get $x_i^{t-pN} - x_i^{t-pN-q} \leq qL + q(2M - \tau + 1) + v_i^{t-pN-q} - M$ so $x_i^{t-pN} \leq pL + q(2M - \tau + 1) + v_i^{t-pN-q} - M \leq (pN + q)(2M - \tau + 1) + v_i^{t-pN-q} - M$ hence by corollary 3.7 $C_i^t$ remains eternally.

Theorem 3.8 provides a way to calculate $\tau_\infty$ within limited steps: $\tau_t$ will not change further if it did not increase within $2N$ steps, and $\tau_t \leq M$. So we simply calculate $2MN$ time steps then $\tau_{2MN}$ must be $\tau_\infty$. Every time step has a computational requirement in proportion to $N$, so this algorithm has a computational requirement in proportion to $N^2$. However, this method is far from elegant and depends on the periodic boundary condition. In next section we will develop a smarter and more precise way to predict the behavior of fronts.

4. Minimal fronts and strictly minimal fronts

Definition 4.1. An $\alpha$-front $C_i^t$ being minimal means that $h_i^t \geq \alpha$ for any $s > t$. If the inequality holds strictly, we say the front is strictly minimal.
Evidently, a $\tau_t$-front is always minimal.

**Lemma 4.2.** If an $\alpha$-front $C_i^t$ is minimal (resp. strictly minimal), then $\phi_t(C_i^t)$ is minimal (resp. strictly minimal). Or more generally, let $C_i^{t-1} = \phi_t(C_i^t)$, then $h_i^s \geq \alpha$ for any $s \geq t$ if $C_i^t$ is minimal; the inequality strictly holds if $C_i^t$ is strictly minimal.

**Proof.** If $C_i^t$ is type B, this is obvious because $\phi_t(C_i^t) = C_i^{t-1}$. Now assume $C_i^t$ to be type A, so $\phi_t(C_i^t) = C_i^{t+1}$. We prove that if there exists an $s \geq t$ such that $h_i^{s+1} = h_i^s + \delta \leq \alpha$, then $h_i^{s+2} \leq \alpha$ (this implies $\delta = \alpha$ when $C_i^t$ is minimal, and leads to a contradiction when $C_i^t$ is strictly minimal):

Since $C_i^t$ is minimal, we have $h_i^r \geq \alpha$ for all $r \geq t$, so $v_i^r \geq \alpha$ for all $r \geq t$. And $C_i^t$ is a type A front so $h_i^{t+1} = v_i^{t+1} \leq M$. Then by lemma 3.1, we have $h_i^r \leq 2M - \alpha$ for all $r \geq t$. In particular, $h_i^s \leq 2M - \alpha$. Now do the following case division which is similar to the one in lemma 3.2:

(i) If $\alpha \leq h_i^s \leq M$:
Since $v_i^r \geq \alpha$ we have $h_i^{s+1} \leq h_i^s + h_i^{s+1} - \alpha = h_i^s + \delta - \alpha \leq h_i^s$, so $v_i^{s+1} = h_i^{s+1}$, then $h_i^{s+2} = v_i^{s+1} \leq h_i^{s+1} + 1 = h_i^s = \delta$.

(ii) If $M \leq h_i^s \leq 2M - \alpha$:
If $v_i^s = h_i^s$ we have $h_i^{s+1} \leq M$. If $v_i^s = \alpha$, then $h_i^{s-1} = v_i^{s-1} - \delta \leq h_i^s + M - \alpha = v_i^s + M - \alpha$, so again $h_i^{s+1} = h_i^s + \alpha - v_i^s \leq M$.

Since $h_i^s \geq M$, we have $v_i^{s+1} = h_i^{s+1}$ and hence $h_i^{s+2} = v_i^{s+1} \leq h_i^{s+1} = \delta$.

**Corollary 4.3.** $\tau_t = \tau_\infty$ if and only if there is no strictly minimal $\tau_t$-front at time $t$.

**Proof.** Necessity (Need the periodic boundary condition): If there is a strictly minimal $\tau_t$-front $C_i^t$, we have $h_i^s > \tau_t$ for all $s \geq t + 1$. Using (ii) of lemma 2.2, we get $h_i^{s-1} > \tau_t$ for all $s \geq t + 1$. Do this repeatedly and note the periodic boundary condition, finally we get $h_i^s > \tau_t$ for all $s \geq t + 2N - 1$ and all $i$. That means $\tau_\infty > \tau_t$.

Sufficiency: If $\tau_\infty > \tau_t$, there exists an $s \geq t$ such that $\tau_{s+1} > \tau_s$. Then all $\tau_s$-fronts in $G_s$ are strictly minimal. Take a $C_i^{s} \in G_s$, the front $\phi_{t+1} \circ \ldots \circ \phi_{s-1} \circ \phi_s(C_i^t) \in F_t$ is strictly minimal by lemma 4.2.

**Notation4.4.** Let $S_i$ be the set consists of all minimal fronts at time $t$. Use $\Phi_i^{t-k}$ to denote the map $\phi_{t-k+1} \circ \ldots \circ \phi_{t-1} \circ \phi_k|_{S_i} : S_i \to S_{t-k}$. $\Phi_i^t$ is understood as the identity map of $S_i$.

**Lemma 4.5.** Suppose $C_i^t$ to be a type A minimal $\alpha$-front. Then we have:

1. $v_i^{t+1} \geq \alpha$ for any $s \geq t$. If $C_i^t$ is strictly minimal, the inequality holds strictly.
2. If $C_i^t$ remains to $C_{i-1}^t$, then $C_i^{t+2}$ is minimal. If $C_i^t$ is strictly minimal, $C_i^{t+2}$ is also strictly minimal.
3. $v_i^{t+2} = h_i^{t+1} = v_i^{t+1}$. 


Proof. (1) Since \( C_i \) is type A, by lemma 3.4 we have \( h_{i+1}^s \geq \alpha \) for any \( s \geq t \). So \( v_{i+1}^t \geq \alpha \) holds for any \( s \geq t \). (2) Since \( h_i^t \geq \alpha \) for any \( s \geq t+1 \), using (ii) of lemma 2.2 we get \( h_{i-1}^s \geq \alpha \) for any \( s \geq t+3 \). (3) Note that \( v_{i+1}^t \geq \alpha \) and this can be checked by simple calculation.

Using lemma 4.5, we find that the condition \( v_{i+1}^{t+k} \geq \alpha \) in lemma 3.4 is always satisfied if we assume \( C_i^{t+k} \) to be minimal. Then using (iii) of lemma 2.2 to estimate the velocities, we get the following generalization of corollary 3.7.

**Corollary 4.6.** Suppose \( C_i \) to be a type A minimal \( \alpha \)-front and assume \( v_{i-j}^t \geq \alpha \) for all \( 0 \leq j \leq k \). Then \( C_i \) remains to \( C_{i-k} \) precisely when \( x_{i}^t - x_{i-j}^t \leq j(2M - \tau_t + 1) + v_{i-j}^t - M \) for all \( 0 \leq j \leq k \).

Proof. Since \( C_i \) is minimal, we have \( v_i^s \geq \alpha \) for all \( s \geq t \). The assumption is \( v_{i-j}^t \geq \alpha \) for all \( 0 \leq j \leq k \). Using (iii) of lemma 2.2 repeatedly we conclude \( v_{i-j}^t \geq \alpha \) for all \( 0 \leq j \leq k \) and all \( s \geq t \). So the premise of lemma 3.4 is satisfied.

Now what happens if a front does not remain? Suppose \( C_i^{t+k} (k \geq 0) \) to be a type A minimal \( \alpha \)-front which does not remain to \( C_{i-1} \). If we have \( v_{i-1}^t \geq \alpha \) for all \( t \leq s \leq t+k \), using remark 3.5 we can get \( h_{i-1}^{t+k} + v_{i-1}^{t+k} = \min(M, h_i^{t+k+2}) \).

If \( \beta = h_{i-1}^{t+k+2} < v_i^{t+k+2} \), we find that \( C_i^{t+k+2} \) is a type A \( \beta \)-front. If not the case, generally we cannot say more without some extra information about the cars before \( C_i^{t+k} \). However, we will show that as for strictly minimal fronts, a prediction indeed can be done since the situation becomes simple enough after a specific time point.

**Lemma 4.7.** Suppose \( C_i \) to be a type A minimal \( \alpha \)-front. Then \( v_{i+1}^{t+1} \leq v_{i}^{t+3} \).

If \( C_i \) is strictly minimal, the inequality strictly holds unless \( v_{i+1}^{t+1} = v_{i}^{t+3} = M \).

Proof. By lemma 4.5, we have \( v_i^{t+2} = h_{i-1}^{t+1} = v_i^t > \alpha \) and \( v_{i+1}^{t+2} \geq \alpha \), so \( h_i^{t+2} = h_{i-1}^{t+1} + v_{i+1}^{t+1} \geq v_i^t + v_{i+1}^{t+1} \), and \( h_i^{t+3} = h_i^{t+1} - (v_i^t - v_{i+1}^{t+2}) \) for any \( 0 \leq i \leq k \). Hence \( v_{i+1}^{t+1} \leq v_{i}^{t+3} \).

**Lemma 4.8.** At time \( t \geq 2M^2 \), a strictly minimal front \( C_i \) must be in one of the following two cases.

1. A type A front with \( v_{i+1}^t < M, v_{i+1}^{t+1} = M \);
   Or a type B front with \( h_i^t < M, v_i^t = v_i^{t+2} = M \).
2. A type A front with \( v_{i+1}^t = M \);
   Or a type B front with \( v_i^t = M \).

Proof. First we prove that if \( C_j^s = \phi_{s+1}(C_i^{s+1}) \) and \( C_k^{s+1} \) is a strictly minimal front, then \( C_j^s \) in case (1) or (2) implies \( C_k^{s+1} \) in case (1) or (2).

- \( C_j^s \) is a type A \( \alpha \)-front in case (1):
  Since \( C_j^s \) is type A, \( C_k^{s+1} = C_j^{s+1} \) is type B and \( h_k^{s+1} = v_{j+1}^s < M, v_{k+1}^{s+1} = v_{j+1}^s = M \) immediately follows. We have \( h_k^{s+2} \geq v_{k+1}^{s+1} = M \), and apply
lemma 4.5 to the type A minimal \( \alpha \)-front \( C^\alpha_j \) we get \( v^{r+2}_{k+1} \geq \alpha \), so \( h^{\gamma+3}_{k} = h^{\alpha+1}_{k} - \alpha - v^{\gamma+2}_{k+1} + M + v^{\alpha+2}_{k+1} = M + v^{\alpha+2}_{k+1} - \alpha \geq M \), that means \( v^{\gamma+3}_{k} = M \). Hence \( C^s_{k+1} \) is in case (1).

- \( C^s_j \) is type B in case (1):
  - If \( j = k \) and \( C^s_{k+1} \) is type B, then \( v^{s+2}_k = v^{s+2}_j = M \), so \( C^s_{k+1} \) is in case (2).
  - If \( k = j - 1 \) and \( C^s_{k+1} \) is type A, we have \( v^{s+1}_{k+1} = v^{s+1}_j = h^s_j < M \), and \( v^{s+2}_{k+1} = v^{s+2}_j = M \), so \( C^s_{k+1} \) is in case (1).

- \( C^s_j \) is type A in case (2):
  - Then \( C^s_{k+1} = C^s_{j+1} \) is type B and we have \( v^{s+2}_k = v^{s+2}_j = v^{s+2}_{j+1} = M \) by lemma 4.5. So \( C^s_{k+1} \) is in case (2).

- \( C^s_j \) is type B in case (2):
  - \( v^{s+1}_j = M \) so \( C^s_{j+1} \) cannot be a front. Hence \( k = j - 1 \) and \( C^s_{k+1} \) is type A, with \( v^{s+1}_{k+1} = v^{s+1}_j = M \). So \( C^s_{k+1} \) is in case (2).

Let \( K_a = \Phi_{t-2M^2}(C^i_t) \). Consider the sequence \( K_a, K_{a+1}, \ldots, K_{a+2M^2} \). If we can find a \( K_a \) in case (1) or (2), then \( C^i_t \) must be in case (1) or (2). Insert a dividing line between \( K_p \) and \( K_{p+1} \) if they are both type B. The total number of dividing lines is less than \( (M-1) \), because if we have \( K_q \) as an \( \alpha \)-front and \( K_{q+1} \) as a \( \beta \)-front, then basically \( \beta \geq \alpha \), and the inequality is strict if \( K_q \) and \( K_{q+1} \) are both type B, which means this both type B case cannot happen more than \( (M-1) \) times. Now these less than \( (M-1) \) dividing lines divide the sequence into less than \( M \) parts, and since the sequence is \( 2M^2 \) long, we can find a part whose length exceeds \( 2M \). Every type A front in this part is followed by a type B front and vice versa, so we finally found a \( 2M \) long sequence \( K_a, K_{a+1}, \ldots, K_{a+2M^2-1} \) with an A,B,A,B,... type pattern. Then apply lemma 4.7 to \( K_a = C^y_x, K_{a+2} = C^y_{x+2}, K_{a+4} = C^y_{x+4}, \ldots \) we found that \( v^{y+1}_x, v^{y+3}_x, v^{y+5}_x, \ldots \) increases to \( M \) at last. Then the type A front \( K_{a+2M^2-2} = C^y_{x-M^2-2} \) with \( v^{y+2M^2-2}_x = M \) is in case (1) if \( v^{y+2M^2-2}_x < M \), or in case (2) if \( v^{y+2M^2-2}_x = M \).

Remark 4.9. Now suppose \( t \geq 2M^2 \), and \( C^1_i \) is a type A strictly minimal \( \alpha \)-front which does not remain to \( C_{i-1} \). Assume we know \( h^{t+2}_{i-1} \) and \( v^{t+2}_{i-1} = \min(M, h^{t+2}_{i-1}) \). Then lemma 4.8 suggests all cases would happen:

(i) \( C^1_i \) is in case (1). Let \( f = h^t_i \).

(a) If \( \beta = h^{t+2}_{i-1} < f \), then \( C_{i-1}^{t+2} \) is a type A \( \beta \)-front, \( v^{t+2}_i = f, v^{t+3}_i = M \).

(b) If \( f \leq h^{t+2}_{i-1} \leq M \), then \( x^{t+3}_{i-1} = x^{t+2}_i - 1, C_{i-1}^{t+3} \) is a type A \( f \)-front, \( v^{t+3}_i = M \).

(c) If \( M < h^{t+2}_{i-1} \leq 2M - f \), then \( x^{t+3}_{i-1} = x^{t+2}_i + M, C_{i-1}^{t+3} \) is a type A \( \gamma \)-front where \( \gamma = h^{t+2}_{i-1} - M + f, v^{t+3}_i = M \).

(d) If \( h^{t+2}_{i-1} \geq 2M - f \), then \( v^{t+2}_i = v^{t+3}_i = v^{t+3}_i = M \).

(ii) \( C^1_i \) is in case (2).

(a) If \( \beta = h^{t+2}_{i-1} < f \), then \( C_{i-1}^{t+2} \) is a type A \( \beta \)-front, \( v^{t+2}_i = M \).
(...)

(b) If \( h_{i-1}^{t+2} \geq M \), then \( v_{i-1}^{t+2} = v_i^{t+2} = M \).

Readers are perhaps worried about how do we know a front remains or not when the estimates of velocities do not hold? In fact this case is avoided well in our algorithm. The mechanism is based on the following lemma, which suggests that if we check from \( 0 \)-fronts to \((M-1)\)-fronts in turn, things will go smoothly.

**Lemma 4.10.** Suppose \( C_i^{t+k}(k \geq 0) \) to be a type A minimal \( \alpha \)-front. If \( C_i^{t-1} \) is NOT a \( \beta \)-front with \( \beta < \alpha \), and \( C_i^{t-1} \) is NOT a type A strictly minimal \( \gamma \)-front for any \( s \geq t+1 \) and any \( \gamma < \alpha \), then \( v_i^{t-1} \geq \alpha \) for all \( s \geq t \).

**Proof.** Let \( \delta = \min \{ h_{i-1}^s \mid s \geq t \} \). Since \( C_i^{t+k} \) is minimal we have \( h_{i-1}^s \geq \alpha \) for all \( s \geq t + k + 2 \) by (ii) of lemma 2.2, so if \( \delta < \alpha \), the minimum \( \delta \) is gained at a time before \( t + k + 2 \). Let \( r = \max \{ s \geq t \mid h_{i-1}^s = \delta \} \) and we find \( C_i^{t-1} \) to be a type A strictly minimal \( \delta \)-front if \( r \geq t + 1 \). As this contradicts our assumption, \( r = t \). This case, since \( C_i^{t-1} \) is not a \( \beta \)-front with \( \beta < \alpha \), we have \( v_i^t \leq v_i^{t-1} = h_{i-1}^t = \delta \), but this implies \( h_{i-1}^{t+1} \leq \delta \) and contradicts the definition of \( r \). So we conclude that \( h_{i-1}^t \geq \alpha \) for all \( s \geq t \) and hence \( v_i^{t-1} \geq \alpha \) for all \( s \geq t + 1 \). Furthermore, \( v_i^{t-1} \geq \alpha \) holds because if \( \epsilon = v_i^{t-1} < \alpha \) then \( C_i^{t-1} \) is a type B \( \epsilon \)-front, contradiction again.

**Remark 4.11.** It is valuable to point out a special case.

First we make an understanding that for a type B \( \alpha \)-front \( C_i^y \) and a \( z \geq 1 \), we say \( C_i^y \) remains to \( C_{x-z} \) in the meaning that \( C_{x-1}^{y+1} \) is a type A \( \alpha \)-front which remains to \( C_{x-z} \).

Now let \( t \geq 2M^2 \), let \( C_i^{t+k}(k \geq 0) \) be a type A minimal \( \alpha \)-front and \( C_i^{t-1} \) a type B strictly minimal \( \beta \)-front with \( \beta < \alpha \). Suppose \( C_i^{t-1} \) remains to \( C_{i-p} \), \( C_i^{t-1} \) is not a type A strictly minimal \( \gamma \)-front for any \( s \geq t+1 \) and any \( \gamma < \alpha \).

We want to know whether \( C_i^{t+k} \) remains to \( C_{i-q} \) for any \( 1 \leq q \leq p^2 \).

First we consider \( C_i^{t-1} \). Lemma 4.8 says there are two cases:

- \( C_i^{t-1} \) satisfies (1) of lemma 4.8.
  - We have \( v_i^{t+2} = M \). So \( C_i^{t+2} \) is not a front. Using lemma 4.10 we get \( v_i^{t-1} \geq \alpha \) for all \( s \geq t + 2 \). Note that \( k \) cannot be 0 because \( v_i^t = M \). For \( k \geq 1 \), using lemma 3.4 we conclude that \( C_i^{t+k} \) remains to \( C_{i-1} \) precisely when \( x_i^{t+k} - x_i^{t+2} - 1 \leq kM - \alpha \).

- \( C_i^{t-1} \) satisfies (2) of lemma 4.8.
  - We have \( v_i^{t+1} = M \), so \( C_i^{t+1} \) is not a front, then \( v_i^{t-1} \geq \alpha \) for all \( s \geq t + 1 \). Using lemma 3.4 we get the condition \( x_i^{t+k} - x_i^{t+1} - 1 \leq (k+1)M - \alpha \).

These two expressions can be unified to \( x_i^{t+k} - x_i^{t+1} - 1 \leq v_i^{t+1} + kM - \alpha \).

Now we consider \( C_{i-q} \). Note that the condition \( v_i^{t-1} \geq \alpha \) for all \( s \geq t + 2 \) (or \( s \geq t + 1 \)) is equivalent to \( h_i^{s-1} \geq \alpha \) for all \( s \geq t + 1 \) (or \( s \geq t \)), so we use (ii) of lemma 2.2 to conclude that the velocity estimates for \( C_{i-q} \) automatically hold. If \( C_i^{t-1} \) satisfies (1) of lemma 4.8, we have \( v_{i-q}^{t+1} = v_{i-1}^{t+1} \) by lemma 4.5, and \( v_{i-q}^{t+2} = M \) by lemma 4.7; if \( C_i^{t-1} \) satisfies (2) of lemma 4.8, we have \( v_{i-q}^{t+2} = M \).
by lemma 4.5. Anyway, the calculation is the same to the $C_{i-1}$ case. Note that $x_{i-\varrho}^{t+k} = x_{i-\varrho}^{t+1} + (q-1)(\beta-1)$, and $x_{i-\varrho}^{t+k+2q-2} = x_{i-\varrho}^{t+k} + (q-1)(\alpha-1)$ if $C_{i-\varrho}^{t+k}$ remains to $C_{i-\varrho+1}$, we get the condition for $C_{i-\varrho}^{t+k}$ to remain to $C_{i-\varrho}$ as:

$$x_{i-\varrho}^{t+k} - x_{i-\varrho+1}^{t+k} - 1 \leq v_{i-\varrho+1}^{t+1} k M - (q-1)(\alpha-\beta).$$

This expression shows that it is automatically determined by the index $q$ whether $C_{i-\varrho}^{t+k}$ remains to $C_{i-\varrho}$. Its analogue to the $\beta = \alpha$ case suggests the following.

Lemma 4.12. Suppose $C_{i-\varrho}^{t+k}$ to be a type A minimal $\alpha$-front and $C_{i-\varrho+1}^{t}$ a type B minimal $\alpha$-front. If $C_{i-\varrho}^{t-1}$ remains to $C_{i-\varrho-1}$, then $C_{i-\varrho+1}^{t+k}$ remains to $C_{i-\varrho+1}$ precisely when $C_{i-\varrho}^{t+k}$ remains to $C_{i-\varrho}$.

Proof. This can be easily showed by lemma 3.3. Note that the estimate of velocities automatically holds, and if $C_{i-\varrho}^{t-1}$ remains to $C_{i-\varrho-1}$, then $h_{i-\varrho-1}^{t+1} = \alpha \leq 2M - \alpha$.

The next lemma is a crucial application of lemma 4.8 which essentially makes the algorithm work.

Lemma 4.13. Let $t \geq 2M^2$. If $C_{i-\varrho}^{t+k}, C_{j-\varrho}^{t+k} \in S_{t+k}$ and $i < j$, then we have $\Phi_{i-\varrho}^{t+k}(C_{i-\varrho}^{t+k}) \neq \Phi_{j-\varrho}^{t+k}(C_{j-\varrho}^{t+k})$ unless the case $j = i + 1$ and $\phi_{i-\varrho}^{t+k}(C_{i-\varrho}^{t+k}) = \phi_{i-\varrho}^{t+k}(C_{j-\varrho}^{t+k})$. More precisely, if a type A front $C_{p}^{t+k}$ and a type B front $C_{p+1}^{t+k}$ at time $s \geq 2M^2$ are both minimal, then there does NOT exist a front $C_{q}^{t+k}$ such that $\phi_{s}^{t+k}(C_{q}^{t+k}) = C_{p+1}^{t+k}$.

Proof. Let $C_{p}^{t+k}$ be an $\alpha$-front and $C_{p+1}^{t+k}$ a $\beta$-front. Then we have $\beta > \alpha$ by definition. Since $C_{p+1}^{t+k}$ is minimal, $h_{p+1}^{t+k} \geq \beta > \alpha$ for all $r \geq s$. Using (ii) of lemma 2.2 we get $h_{p+1}^{t+k} \geq \alpha$ also holds because $C_{p}^{t+k}$ is type A. Hence $C_{p}^{t+k}$ is a strictly minimal front which must be in case (1) or (2) of lemma 4.8. $v_{p+1}^{t+k} < M$ because $C_{p+1}^{t+k}$ is a front, so it is case (1). Then we find $C_{p+1}^{t+k}$ to be a type B front with $v_{p+1}^{t+k} + 1 = M$. That means there is no front can be mapped to $C_{p+1}^{t+k}$.

According to this lemma, if we restrict $\Phi_{t+k}^{t+k}$ to $G_{t+k}$ or to the set consists of all type A minimal fronts at time $t+k$, then we get a one-to-one map. Moreover, the reverse of this map can be searched like this:

For minimal front $C_{i}^{t}$, if $C_{i}^{t}$ is type A, then we next go to $C_{i+1}^{t+1}$. If $C_{i}^{t}$ is type B, then we go to $C_{i+1}^{t+1}$ when $C_{i+1}^{t}$ is a type A front, or to $C_{i+1}^{t+1}$ when not.

Notation 4.14. Notation for us to describe the algorithm.

- If $C_{j}^{t} = \Phi_{i}^{t}(C_{i}^{t})$, we say $C_{i}^{t}$ is mapped to $C_{j}^{t}$, or $C_{j}^{t}$ is an image of $C_{i}^{t}$.
- For a minimal front $C_{i}^{t}$, we say $P(C_{i}^{t})$ is the domain of $C_{i}^{t}$ and define it as $P(C_{i}^{t}) = \{ C_{j}^{t} | C_{j}^{t} = \Phi_{i}^{t}(C_{i}^{t}), 0 \leq k \leq s \}$. Give a partial order “$<” between two minimal fronts as $C_{i}^{t} \prec C_{j}^{t} \iff P(C_{i}^{t}) \subseteq P(C_{j}^{t})$. Note that $C_{i}^{t} \prec C_{j}^{t}$ is equivalent to $C_{i}^{t} \in P(C_{j}^{t})$. 

17
First of all, calculate Algorithm 4.15. prove its correctness, then show the details of the subroutine.

Now we are ready to see the algorithm. We first show the main program and case.

Stationary state is then entirely understood. Now we consider the $\infty$ elements of $\Lambda$. if there is no $\alpha Y$ of $Y$ such that $\infty \tau \in \alpha < \tau$. Obviously the elements of $\Lambda$ are minimal, 0 is complete to $X_0$, and there is no $\alpha$-front in $Y_0$ with $\alpha < 0$. By induction, we can assume that the elements of $\Lambda$ are minimal, $Y_0$ is complete to $X_0$, and there is no $\alpha$-front in $Y_0$ such that $\alpha < \tau$. If $\infty \in D_\tau$, of course $\tau_\infty = \tau$, and by the completeness of $Y_\tau$, we can calculate all $\tau_\infty$-fronts at time $t \gg 0$ from $R$. Those $\tau_\infty$-fronts finally connect properly to their preceding fronts by theorem 2.14, so the final stationary state is then entirely understood. Now we consider the $\infty \notin D_\tau$ case.

**Algorithm 4.15.** First of all, calculate $2M^2$ time steps and reset the time to 0.

- let $Y_0 = F_0$
- if $F_0 = \emptyset$ then it is a free or uniform state, end the program.
- for $\tau = 0$ to $M - 1$
  - let $\Lambda_\tau = \{\tau$-fronts in $Y_\tau\}$.
  - subroutine:
    - calculate $D_\tau = \{d(C_i^\tau) | C_i^\tau \in \Lambda_\tau\}$. If $\infty \in D_\tau$, output $\tau_\infty = \tau$ and $R = \{C_i^\tau \in \Lambda_\tau | d(C_i^\tau) = \infty\}$, end the program. If $\infty \notin D_\tau$, find the maximal elements of $D_\tau$ in the partial order "$\prec$", put these elements into $Z_\tau$, and calculate $Y_\tau' = Y_\tau \setminus \bigcup_{C_i^\tau \in Z_\tau} P(C_i^\tau)$, $E_\tau = \{e(C_i^\tau) | C_i^\tau \in Z_\tau\}$.
  - let $Y_{\tau + 1} = Y_{\tau}' \cup E_\tau$.
- conclude $\tau_\infty = M$, end the program.

**Proof of Correctness.** Let $X_0 = X$ and $X_{\tau + 1} = X_\tau \setminus \bigcup_{C_i^\tau \in Z_\tau} P(C_i^\tau)$ if $\infty \notin D_\tau$. Obviously the elements of $\Lambda_0$ are minimal, $Y_0$ is complete to $X_0$, and there is no $\alpha$-front in $Y_0$ with $\alpha < 0$. By induction, we can assume that the elements of $\Lambda_\tau$ are minimal, $Y_\tau$ is complete to $X_\tau$, and there is no $\alpha$-front in $Y_\tau$ such that $\alpha < \tau$. If $\infty \in D_\tau$, of course $\tau_\infty = \tau$, and by the completeness of $Y_\tau$, we can calculate all $\tau_\infty$-fronts at time $t \gg 0$ from $R$. Those $\tau_\infty$-fronts finally connect properly to their preceding fronts by theorem 2.14, so the final stationary state is then entirely understood. Now we consider the $\infty \notin D_\tau$ case.
(i) There is no type A minimal $\beta$-front with $\beta < \tau$ exists in $X_\tau$: 
If we have a type A minimal $\beta$-front in $X_\tau$, it is mapped to an $\alpha$-front with $\alpha \leq \beta$ in $Y_\tau$ since $Y_\tau$ is complete to $X_\tau$. However, there is no $\alpha$-front in $Y_\tau$ such that $\alpha < \tau$.

(ii) $\tau_\infty > \tau$: 
Since $\infty \notin D_\tau$, we can select a time point $s > \max\{t | C^t_1 \in Z_\tau\}$, then there is no type A minimal $\beta$-front with $\beta < \tau$ exists after time $s$ by (i). We will show $\beta$ cannot be $\tau$ either. That is because if such a $\beta$-front exists, it is mapped to a $C^j_\tau$ in $\Lambda_\tau$, and so $C^j_\tau$ remains to a time after $s$. That contradicts the definition of $s$.

(iii) Elements of $Z_\tau$ are strictly minimal: 
Take a $C^i_1 \in Z_\tau$ and assume $\delta = \min\{h^i_s | s > t\} \leq \tau$. The elements of $\Lambda_\tau$ are minimal so by lemma 4.5 $C^i_1$ is also minimal. Then $\delta = \tau$. Since $\tau_\infty > \tau$, we can take $r = \max\{s > t | h^i_s = \delta\}$ and find $C^i_r$ to be a type A minimal $\tau$-front. This contradicts the definition of $Z_\tau$, for we have $C^i_1 < C^i_r$, and $C^i_1$ is mapped to a front $C^i_p$ in $\Lambda_\tau$, then $d(C^i_p) > C^i_r > C^i_1$.

(iv) For any $C^i_s \in Z_\tau$, let $s$ be such that $C^i_s-1 \in X_\tau$ and $C^i_s-1 \notin X_\tau$, then $v^s_{r-1} = M$ for all $s + 1 \leq r \leq t$ (i.e. the “left border” of $P(C^i_1)$ is coated by a “velocity $M$ wall”): 
If $t = 0$, there is nothing to prove. If $t > 0$, then $C^i_t-1$ is a type A minimal $\tau$-front. If $s = 0$, then $C^i_s-1 \in Y_\tau$, so $C^i_s-1$ is not an $\alpha$-front with $\alpha < \tau$. Then by lemma 4.10 we have $v^s_{r-1} \geq \tau$ for all $r \geq s$, and since $C^i_t-1$ does not remain to $C^i_s-1$, we have $v^s_{r-1} = M$ for all $s + 1 \leq r \leq t$ by remark 3.5. If $s > 0$, $C^i_s-1$ must be a type B strictly minimal $\beta$-front with $\beta < \tau$ by (iii). Since $C^i_t-1 \in X_\tau$, we have $t - 1 \geq s - 1$ (the $t = s - 1$ or $t = s - 2$ case is avoided since $C^i_s-1$ is strictly minimal and by lemma 4.8). So as we have discussed in remark 4.11, we also have the velocity estimates and then can use remark 3.5.

(v) $Y_\tau+i$ is complete to $X_\tau+i$: 
$Y_\tau+i$ is complete to $X_\tau$ and so complete to $X_\tau+i$. Then for any minimal type A front $C^j_1 \in X_\tau+i$, there exists a $C^j_k \in Y_\tau$ such that $C^j_k = \Phi^j(Y_\tau)$. If $C^j_k \in Y_\tau+i$ there is nothing to prove. If $C^j_k \notin Y_\tau+i$, since $Y_\tau+i \supseteq Y_\tau \cup \bigcup_{C^j_t \in Z_\tau} P(C^j_t)$, there exists a $C^j_t \in Z_\tau$ such that $C^j_t \notin P(C^j_k)$. On the other hand $C^j_s \notin \Phi^j(Y_\tau)$, so we can find a $u \leq r$ such that $\Phi^j_u(C^j_s) \notin \Phi^j_u(Y_\tau)$ and $\Phi^j_u+1(C^j_s_i) \notin \Phi^j_u+1(Y_\tau)$. Let $C^u_i$ denote $\Phi^j_u(C^j_s_i)$. If $\Phi^j_u+1(C^j_s_i) = C^j_t-1$ (i.e. $C^j_t$ is at the “left border” of $P(C^j_t)$), then $l = i$, so $C^u_i$ must be $C^j_t$ by (iv). If $\Phi^j_u+1(C^j_s_i) = C^j_t$ (i.e. $C^j_t$ is at the “right border” of $P(C^j_t)$), then $C^u_i$ is an image of $C^t_i$, so by lemma 4.13, again $C^u_i$ must be $C^j_t$. Anyway $C^u_i$ is an image of $C^j_t$, and now $e(C^j_t_i)$ is the first type A front mapped to $C^j_t$, so we conclude that $e(C^j_t_i)$ is an image of $C^j_t_i$.

(vi) There is no $\alpha$-front with $\alpha < \tau + 1$ in $Y_\tau+i$: 
Because $\Lambda_\tau \subseteq \bigcup_{C^j_t \in Z_\tau} P(C^j_t)$, so there is no $\tau$-front in $Y_\tau+i$; and every element in $E_\tau$ is a type A $\gamma$-front with $\gamma > \tau$.

(vii) Elements of $\Lambda_\tau+i$ are minimal: 
Take a $C^j_t \in \Lambda_\tau+i$ and assume $\delta = \min\{h^j_s | s > t\} \leq \tau$. Since $\tau_\infty > \tau,$
we can take \( r = \max\{s > t \mid h_i^s = \delta\} \) and find \( C_i^r \) to be a type A minimal \( \delta \)-front. Since \( Y_{r+1} \) is complete to \( X_{r+1} \), \( C_i^r \) is mapped to an \( \epsilon \)-front with \( \epsilon \leq \delta \) in \( Y_{r+1} \). However there is no such an \( \epsilon \)-front by (vi).

Now we have accomplished the induction and finished the proof.

In fact, any \( \alpha \)-front lies in \( P(C_i^t)(C_i^t \in Z_r) \) with \( \alpha > \tau \) cannot be minimal, so this algorithm actually calculates the behavior of all minimal fronts.

Now we are going to see the details of the subroutine.

- **data structure**: The SET structure is a collection of \( C_i^t \)'s, we can delete (resp. add) an element from (resp. into) the collection. Also, it has an index \( i \), for a SET \( S \), we can use the method \( S.get(i) \) to get an \( i \)th car \( C_i^t \) in \( S \). If there is no \( i \)th car in \( S \), the get method return null. The method \( S.delete(i) \) deletes all \( i \)th cars in \( S \). Using “for each \( C_i^t \in S \)” we can enumerate all \( C_i^t \)'s in \( S \) from small \( i \) to large \( i \), when \( i \) is the same from small \( t \) to large \( t \), in turn. Let \( \Lambda_r \) and \( Y_r \) have the SET structure.

- **initialization**: keep an array \((x_i, v_i, t_i)\) with index \( i \) in the memory, initialize \((x_i, v_i, t_i)\) to be \((x_i^0, v_i^0, 0)\) for all \( i \).

- **subroutine begin**

  - let \( i_{prev} = \text{null}, C_{prev} = \text{null} \)

  - To avoid the technical difficulties due to the periodic boundary condition, here we normalize the index \( i \) of all \( C_i^t \) in \( \Lambda_r \) to \( 0 \leq i \leq N - 1 \). Then we double \( \Lambda_r \) in the meaning that if \( C_i^t \in \Lambda_r \) with \( 0 \leq i \leq N - 1 \), then \( C_{i+N}^t \in \Lambda_r \).

  - for each \( C_i^t \in \Lambda_r \)

    - let \((j, s, x) = (i, t, x_i^t)\)

    - if \( i = i_{prev} \), then:

      - if \( d(C_{prev}) = \infty \) then conclude \( d(C_i^t) = \infty \), next \( C_i^t \).

      - else, we have \( d(C_{prev}) = C_i^{t-p} \), delete \( C_{prev} \) from \( \Lambda_r \), redefine \((j, s, x) = (i-p, t+2p, x_i^{t+2p}+p(\tau-1))\). \( C_i^t \) remains to \( C_i^{t-p} \), verified by lemma 4.12

    - redefine \( i_{prev} = i \), \( C_{prev} = C_i^t \)

    - if \( C_i^t \) is type B, then: (this case, \( t \) must be 0 by definition of \( Y_r \))

      - if \( C_i^t \) does not remain to \( C_{i-1} \), then conclude \( d(C_i^t) = C_i^{t-1} \), next \( C_i^t \).

      - else, redefine \((j, s, x) = (i-1, t+1, x_i^{t+1}+1)\)

    - while \( x - x_{j-1} - 1 \leq v_{j-1} + (s - t_{j-1} + 1)M - \tau \) do:

      - \( C_i^t \) remains to \( C_{j-1} \)

      * If \( \Lambda_r.get(j-1) \) is not null then:

        - we have \( \Lambda_r.get(j-1) = C_{j-1}^w \).

        - if \( d(C_{j-1}^w) = \infty \) then conclude \( d(C_i^t) = \infty \), next \( C_i^t \).

        - else, we have \( d(C_{j-1}^w) = C_{j-1}^w \), delete \( C_{j-1}^w \) from \( \Lambda_r \), redefine \((j, s, x) = (j-q, s+2q, x-q(\tau-1))\).
* else redefine \((j, s, x) = (j - 1, s + 2, x + \tau - 1)\)
  * if \(j \leq i - N\), then conclude \(d(C^i_j) = \infty\), next \(C^i_j\)
    - conclude \(d(C^i_j) = C^{i+1}_j\)

- Now we can “undouble” \(\Lambda_\tau\) by delete all \(C^i_j \in \Lambda_\tau\) whose index \(i \geq N\).
- \(D_\tau\) is obtained. if \(\infty \in D_\tau\), then output \(\tau_{\infty} = \tau\) and \(R = \Lambda_\tau\), end the program. else, \(Z_\tau = \{d(C^i_j) | C^i_j \in \Lambda_\tau\}\).
- For every \(C^i_j \in Z_\tau\), calculate \(e(C^i_j)\). If \(t = 0\), we simply check the data of time 0. If \(t > 0\) then \(C^{i-1}_j\) is a type A strictly minimal \(\tau\)-front, and we have done the calculation in remark 4.9.
- Refresh \((x_i, v_i, t_i)\) and calculate \(Y^*_\tau\): For every \(C^i_j \in \Lambda_\tau\) with \(d(C^i_j) = C^u_{i-p}\), if \(C^i_j\) is type A, redefine \((x_{i-q}, v_{i-q}, t_{i-q}) = (x^i_j + 2\tau + q(\tau - 1), v^i_j + 2, t + 2 + 2q)\) and \(Y_\tau.\text{delete}(i - q)\) for all \(0 \leq q \leq p\); if \(C^i_j\) is type B, redefine \((x_{i-q}, v_{i-q}, t_{i-q}) = (x^i_j + \tau + q(\tau - 1), v^i_j + 1, t + 2 + 2q)\) and \(Y_\tau.\text{delete}(i - q)\) for all \(0 \leq q \leq p\).
- The refresh of \((x_i, v_i, t_i)\) is verified by the calculation in remark 4.11, and now the modified \(Y_\tau\) is just \(Y^*_\tau\).
- 

```
subroutine end
```

About the computational requirement of this algorithm, note that \(#E_\tau \leq \#D_\tau \leq \#\Lambda_\tau\), so \(#Y^{t+1}_\tau \leq \#Y_{\tau} \leq \#Y_{0} \leq N\). One time the subroutine runs, the \textbf{while} do loop runs less than \(4N\) times, because once we found \(C^i_j\) remains to \(C^i_{i-p}\), the \([i - p, i - 1]\) interval is then skipped afterward. Similarly, the \((x_i, v_i, t_i)\) refresh loop runs less than \(N\) times, because for every two \(C^i_j, C^u_{j-q}\) in \((\text{modified})\Lambda_\tau\) with \(d(C^i_j) = C^u_{i-p}, d(C^u_{j-q}) = C^u_{j-q}\), the interval \([i - p, i]\) and \([j - q, j]\) are disjoint. So we conclude that this is an \(O(N)\) algorithm. Besides, corollary 4.6 and remark 4.11 just can be used for optimizing the subroutine.

5. Summary and further discussion

We have classified all stationary states of Eq.(2), and proved that every state finally evolves to a stationary state if we assume the periodic boundary condition. The results about the “stability” of a \(\tau\)-front show that the smaller \(\tau\) is, the easier the front remains. Tools to investigate the detailed behavior of fronts are developed, briefly speaking, after a specific time point, strictly minimal fronts emerge in a clear shape and then dominate the time evolution of cars. We think these results grasped the main characters of model (1).

All these characters are described with the concept “front”, which focus on the non-trivial behavior of cars in an accelerating process. We think this is an important concept for us to analyze traffic CA models in Lagrange form. For example, there is another widely used (such as in the NS model) effect which is considered to be associated with the inertia of cars. It requires that
\( v_i^t \leq v_i^{t-1} + 1 \), i.e. the acceleration cannot exceed 1. This effect however does not result in multiple branches in the fundamental diagram, and we find that if the term \( h_i^{t-1} \) of model (1) is replaced by \( v_i^{t-1} + 1 \), the process for successive cars with headway \( \tau \) to accelerate from velocity \( \tau \) to the free speed \( M \) always leads to successive cars with headway \( M \), but not the characteristic headway \( 2M - \tau \). We think analysis focused on such kind of behavior can reveal some nature of a traffic CA.

Now we are going to propose a generalization of model (1) which takes in a driver’s anticipation. A kind of generalization is proposed by Nishinari,[12] but with too many terms to avoid a collision. Our generalized equation is written as:

\[
\begin{align*}
    u_i^t &= \min(M, h_i^t, h_i^{t-1}) \\
u_i^t &= \min(M, u_i^t + [\alpha_i u_{i+1}^t + \beta_i])
\end{align*}
\]

\( h_i^t \) and \( v_i^t \) are as above. \( \alpha_i \) and \( \beta_i \) are real numbers such that \( 0 \leq \alpha_i \leq 1, 0 \leq \beta_i < 1 \). \([ \cdot \] is known as the Gauss function, \([ x \] represents the maximal integer which does not exceed \( x \). Let \( \alpha_i = \beta_i = 0 \) for all \( i \) we get Eq.(2).

**Proposition 5.1.** There is no collision in this generalized model.

**Proof.** What to prove is that \( h_i^{t+1} = h_i^t - v_i^t + v_{i+1}^t \) cannot be negative. If \( v_{i+1}^t = M \), this is obvious. So we assume \( v_{i+1}^t = u_{i+1}^t + [\alpha_v u_{i+2}^t + \beta_{i+1}] \geq u_{i+1}^t \). 
\( h_i^t \geq u_i^t \) by Eq.(3), \( v_i^t \leq u_i^t + [\alpha_i u_{i+1}^t + \beta_i] \) by Eq.(4). So it is enough to prove 
\( u_i^t - (u_i^t + [\alpha_i u_{i+1}^t + \beta_i]) + u_{i+1}^t \geq 0 \). Note that \( [\alpha_i u_{i+1}^t + \beta_i] \leq [u_{i+1}^t + \beta_i] = u_{i+1}^t \), so the inequality follows.

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Note that the parameter \( \alpha_i \) and \( \beta_i \) can be chosen differently for each \( i \), or vary randomly in the time evolution. The NS model introduced a random braking which is known to be responsible for spontaneous jam formation, however it is likely to find a spontaneous jam formation in this generalized model when we set each \( \alpha_i \) and \( \beta_i \) differently to get a but deterministic model. When \( \alpha_i = \alpha, \beta_i = \beta \) for all \( i \), we have an analogue to lemma 2.2 as the following:

**Proposition 5.2.** When \( \alpha_i = \alpha, \beta_i = \beta \) for all \( i \), the following inequalities hold:

\[
\begin{align*}
    (i) & \quad h_i^{t+1} \geq \min(h_i^t, u_{i+1}^t, u_{i+2}^t) \\
    (ii) & \quad h_i^{t+1} \geq \min(M, h_i^t, h_{i+1}^t, h_{i+2}^t, h_{i+1}^{t-1}, h_{i+2}^{t-1}) \\
    (iii) & \quad u_i^{t+1} \geq \min(u_i^t, u_{i+1}^t, u_{i+2}^t)
\end{align*}
\]

**Proof.** (i) If \( u_{i+1}^t = M \), then \( h_i^{t+1} \geq h_i^t \). Now we assume \( v_{i+1}^t = [\alpha u_{i+2}^t + \beta] + u_{i+1}^t \). Then

\[
\begin{align*}
    h_i^{t+1} &= h_i^t + v_{i+1}^t - v_i^t \\
    &\geq h_i^t + ([\alpha u_{i+2}^t + \beta] + u_{i+1}^t) - ([\alpha u_{i+1}^t + \beta] + u_i^t) \\
    &\geq h_i^t + [\alpha (u_{i+2}^t - u_{i+1}^t)] + u_{i+1}^t - u_i^t \quad \text{(for } [a] - [b] \geq [a - b])
\end{align*}
\]

22
\[
\begin{align*}
\geq \ & [\alpha (u_{t+2}^i - u_{t+1}^i) + u_{t+1}^i] \quad \text{(for } h_t^i \geq u_t^i) \\
= \ & [\alpha u_{t+2}^i + (1 - \alpha) u_{t+1}^i] \\
\geq \ & \min(u_{t+1}^i, u_{t+2}^i)
\end{align*}
\]

(ii) Just substitute Eq.(3) into (i).

(iii) \[u_{t+1}^i = \min(M, h_{t+1}^i, h_t^i) \geq \min(M, u_{t+1}^i, u_{t+2}^i, h_t^i) \geq \min(u_t^i, u_{t+1}^i, u_{t+2}^i).\]

(ii)(iii) of proposition 5.2 can be understood as that there is no spontaneous jam formation when \(\alpha_i\) and \(\beta_i\) are set to be the same for all \(i\).

The Gauss function here is introduced only for changing the real number into integer, however it certainly makes some differences. It is not clear yet whether this model agrees with the realistic traffic.

References


